

# GENERALIZED EGOROFF'S THEOREM

MIROSLAV REPICKÝ

ABSTRACT. This note is closely related to the paper [8] and it presents slight improvements of its results. Theorem 1.13 shows a connection with Galois-Tukey embeddings; Corollary 1.14 presents another inequality which is dual to the previously known one; Corollary 3.5 shows that there is no distinction between positive outer measure and full outer measure in the given context; and Corollary 4.3 unifies the known counterexamples.

**Notation.** For  $x, y \in {}^\omega\mathbb{R}$ ,  $x \leq y$  means that  $x(n) \leq y(n)$  for all  $n \in \omega$  and  $x \leq^* y$  means that  $x(n) \leq y(n)$  for all but finitely many  $n \in \omega$  (shortly written by  $\forall^\infty n$ ). The set of all sequences of real numbers converging to 0 is denoted by  $c_0$ ;  $c_0^+$  is the set of all positive sequences from  $c_0$ . Clearly,  $c_0 \subseteq {}^\omega\mathbb{R}$  and  ${}^\omega\omega \subseteq {}^\omega\mathbb{R}$ ; the restriction of  $\leq^*$  to  $c_0$  and  ${}^\omega\omega$  will be denoted by the same symbol. The  $\sigma$ -ideal on  ${}^\omega\omega$  generated by compact subsets of  ${}^\omega\omega$  is denoted by  $\mathcal{K}_\sigma$ . It is known that a set  $A \subseteq {}^\omega\omega$  is in  $\mathcal{K}_\sigma$  if and only if there is  $y \in {}^\omega\omega$  such that  $x \leq^* y$  for all  $x \in A$ . The values of the composition of functions  $f \circ g$  we compute by  $f \circ g(x) = f(g(x))$ .

If  $\mathcal{I} \subseteq \mathcal{P}(X)$ , then

$$\begin{aligned} \text{add}(\mathcal{I}) &= \min\{\mathcal{F} \subseteq \mathcal{I} : \bigcup \mathcal{F} \notin \mathcal{I}\}, \\ \text{non}(\mathcal{I}) &= \min\{|Y| : Y \in \mathcal{P}(X) \setminus \mathcal{I}\}, \\ \text{cov}(\mathcal{I}) &= \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{F} = X\}, \\ \text{cof}(\mathcal{I}) &= \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \text{ and } (\forall Y \in \mathcal{I})(\exists Z \in \mathcal{F}) Y \subseteq Z\}. \end{aligned}$$

Let us recall that  $\mathfrak{b} = \text{add}(\mathcal{K}_\sigma) = \text{non}(\mathcal{K}_\sigma)$  and  $\mathfrak{d} = \text{cov}(\mathcal{K}_\sigma) = \text{cof}(\mathcal{K}_\sigma)$  is the unbounding number and the dominating number, respectively.

## 1. GENERALIZED EGOROFF'S THEOREM

Let us recall that a sequence of real-valued functions  $\langle f_n : n \in \omega \rangle$  *quasinormally* converges to  $f$  on  $X$  (shortly written by  $f_n \xrightarrow{\text{QN}} f$ ), if  $(\exists \varepsilon \in c_0)(\forall x \in X)(\forall^\infty n \in \omega) |f_n(x) - f(x)| < \varepsilon(n)$ . The uniform convergence is denoted as usual by  $\rightrightarrows$ . Let us recall that the convergence on  $X$  is quasinormal if and only if  $X$  can be partitioned into countably many sets on which the convergence is uniform. Conversely, if  $X$  can be partitioned to  $< \mathfrak{b}$  sets on which the convergence is uniform, then the convergence is quasinormal on  $X$  (see [4]).

**Definition 1.1.** Let  $\mathcal{F} \subseteq \mathcal{P}(X)$ . Let us consider the following statements:

$E(\mathcal{F})$  Given a sequence  $\langle f_n : n \in \omega \rangle$  of functions  $f_n : X \rightarrow \mathbb{R}$  converging pointwise to 0 there exists a set  $A \in \mathcal{F}$  such that  $f_n \xrightarrow{\text{QN}} 0$  on  $A$ .

---

2000 *Mathematics Subject Classification.* Primary 03E17; Secondary 03E35 28A20 40A30.

*Key words and phrases.* Egoroff's theorem, measure, category, cardinal invariants, Galois-Tukey embeddings.

The work has been supported by grant of Slovak Grant Agency VEGA 1/3002/26.

$E^*(\mathcal{F})$  Given a sequence  $\langle f_n : n \in \omega \rangle$  of functions  $f_n : X \rightarrow \mathbb{R}$  converging pointwise to 0 there exists a set  $A \in \mathcal{F}$  such that  $f_n \rightrightarrows 0$  on  $A$ .

*Remark 1.2.* (a) The sequence of functions  $f_n : X \rightarrow \mathbb{R}$ ,  $n \in \omega$  that pointwise converges to 0 on  $X$  can be identified with a single function  $F : X \rightarrow c_0$  by setting  $F(x)(n) = f_n(x)$  (see [8]). Sometimes when we will speak about  $F$  we will mean the sequence  $\langle f_n : n \in \omega \rangle$ . This convention will be applied mainly in the context of uniform and quasinormal convergence. In particular  $E(\mathcal{F})$  can be rephrased as follows: For every  $F : X \rightarrow c_0$  there exists  $A \in \mathcal{F}$  such that  $F \xrightarrow{\text{QN}} 0$  on  $A$ .

(b) The definition of  $E(\mathcal{F})$  can be equivalently restricted to monotone systems of functions because  $f_n \xrightarrow{\text{QN}} 0$  if and only if  $f'_n \xrightarrow{\text{QN}} 0$ , where  $f'_n(x) = \sup\{|f_k(x)| : k \geq n\}$ .

(c) If  $\overline{\mathcal{F}}$  is the closure of  $\mathcal{F}$  under supersets, then  $E(\overline{\mathcal{F}})$  is equivalent to  $E(\mathcal{F})$ , and  $E^*(\overline{\mathcal{F}})$  is equivalent to  $E^*(\mathcal{F})$ .

**Definition 1.3** ([8]). We define  $\alpha : c_0 \rightarrow {}^\omega\omega$ ,  $\beta : {}^\omega\omega \rightarrow c_0$ ,  $\theta : {}^\omega\omega \rightarrow {}^\omega\omega$ , and  $\rightarrow : {}^\omega\omega \rightarrow {}^\omega\omega$  by

$$\begin{aligned}\alpha(x)(n) &= \min\{m \in \omega : (\forall k \geq m) x(k) \leq 2^{-n}\}, \\ \beta(y)(k) &= \min\{2^{-n} : \theta(y)(n) \leq k\}, \\ \theta(y)(n) &= \max(\{n\} \cup \{y(k) : k < n\}), \\ y^\rightarrow(n) &= \max\{y(0), y(n+1)\}\end{aligned}$$

for  $x \in c_0$ ,  $y \in {}^\omega\omega$ , and  $n, k \in \omega$ .

**Lemma 1.4.**  $\alpha \circ \beta = \theta$ .

*Proof.*  $\alpha(\beta(y))(n) = \min\{m : (\forall k \geq m) \beta(y)(k) \leq 2^{-n}\} = \min\{m : \beta(y)(m) \leq 2^{-n}\} = \min\{m : \theta(y)(n) \leq m\} = \theta(y)(n)$ .  $\square$

**Lemma 1.5.** If  $Y \subseteq {}^\omega\omega$ , then  $Y \in \mathcal{K}_\sigma$  if and only if  $\theta(Y) \in \mathcal{K}_\sigma$ .

*Proof.* If  $Y \in \mathcal{K}_\sigma$ , then  $\theta(Y) \in \mathcal{K}_\sigma$  because  $\theta$  is continuous. Conversely, let us assume that  $\theta(Y) \in \mathcal{K}_\sigma$ , i.e., there is  $z \in {}^\omega\omega$  such that  $\theta(y) \leq^* z$  for all  $y \in Y$ . Then  $y \leq^* z^\rightarrow$  for all  $y \in Y$  and hence  $Y \in \mathcal{K}_\sigma$ .  $\square$

**Lemma 1.6.**

- (1)  $(\forall x \in c_0)(\exists y \in {}^\omega\omega) x \leq^* \beta(y)$ ; in fact  $x \leq \beta(\alpha(2x))$ .
- (2)  $(\forall x \in c_0)(\forall z \in c_0^+ \text{ monotone}) x \leq^* z \Rightarrow \alpha(x) \leq^* \alpha(z) \Rightarrow x \leq^* 2z$ .
- (3)  $y \leq \theta(y^\rightarrow)$  for  $y \in {}^\omega\omega$ .
- (4)  $(\forall y \in {}^\omega\omega)(\forall z \in {}^\omega\omega \text{ monotone unbounded}) y \leq^* z \Leftrightarrow \beta(y) \leq^* \beta(z)$ .  $\square$

*Proof.* (1)  $\beta(\alpha(2x))(k) = \min\{2^{-n} : \theta(\alpha(2x))(n) \leq k\} = \min\{2^{-n} : n \leq k \text{ and } \alpha(2x)(n-1) \leq k\} \geq \min\{2^{-n} : (\forall i \geq k) 2x(i) \leq 2^{-(n-1)}\} \geq x(k)$ .

(2) As  $z \in c_0^+$ , then  $x \leq^* z$  if and only if there is  $n_0$  such that  $z(k) \leq 2^{-n_0}$  implies  $x(k) \leq z(k)$  and then  $\alpha(x)(n) \leq \alpha(z)(n)$  for  $n \geq n_0$ . If  $\alpha(x)(n) \leq \alpha(z)(n)$  for  $n \geq n_0$ , then as  $z$  is monotone, for every  $k \in \omega$  and every  $n \geq n_0$ ,  $z(k) \leq 2^{-n}$  implies  $x(k) \leq 2^{-n}$ . Hence  $2^{-n-1} < z(k) \leq 2^{-n}$  implies  $x(k) \leq 2^{-n} < 2z(k)$ . Consequently,  $x \leq^* 2z$ .

(4)  $y \leq^* z$  implies  $\theta(y) \leq^* \theta(z)$ . Let  $n_0$  be such that  $\theta(y)(n) \leq \theta(z)(n)$  for all  $n \geq n_0$ . Then for  $k \geq \theta(z)(n_0)$ ,  $\{n : \theta(y)(n) \leq k\} \supseteq \{n : \theta(z)(n) \leq k\}$  and so  $\beta(y)(k) \leq \beta(z)(k)$ . Conversely, if  $\beta(y) \leq^* \beta(z)$ , then there is  $k_0$  such that  $\theta(z)(n) = k$  implies  $\theta(y)(n) \leq k$  for  $k \geq k_0$  and so  $\theta(y) \leq^* \theta(z)$  (notice that  $\theta(z)$  is

monotone unbounded). Now, as  $z$  is monotone unbounded we can easily see that  $y \leq^* z$ .  $\square$

**Lemma 1.7.** *Let  $F : X \rightarrow c_0$  and  $G : X \rightarrow {}^\omega\omega$ .*

- (a)  $F \xrightarrow{QN} 0$  on  $X$  if and only if  $(\alpha \circ F)(X) \in \mathcal{K}_\sigma$ .
- (b)  $G(X) \in \mathcal{K}_\sigma$  if and only if  $\beta \circ G \xrightarrow{QN} 0$  on  $X$ .

*Proof.* In the following equivalences the assertions of Lemma 1.6 are applied:

$$\begin{aligned} F \xrightarrow{QN} 0 \text{ on } X &\Leftrightarrow (\exists \varepsilon \in c_0)(\forall x \in X) F(x) \leq^* \varepsilon \\ &\Leftrightarrow (\exists y \in {}^\omega\omega)(\forall x \in X) F(x) \leq^* \beta(y) && \text{by (1),} \\ &\Leftrightarrow (\exists y \in {}^\omega\omega)(\forall x \in X) \alpha \circ F(x) \leq^* \theta(y) && \text{by (2) and Lemma 1.4,} \\ &\Leftrightarrow (\exists y \in {}^\omega\omega)(\forall x \in X) \alpha \circ F(x) \leq^* y && \text{backward use (3).} \end{aligned}$$

$$\begin{aligned} \beta \circ G \xrightarrow{QN} 0 \text{ on } X &\Leftrightarrow (\exists y \in {}^\omega\omega)(\forall x \in X) \beta \circ G(x) \leq^* \beta(y) && \text{by (1),} \\ &\Leftrightarrow (\exists y \in {}^\omega\omega)(\forall x \in X) G(x) \leq^* y && \text{by (4).} \quad \square \end{aligned}$$

**Theorem 1.8** ([8]). *Let  $\mathcal{F} \subseteq \mathcal{P}(X)$ . The following conditions are equivalent:*

- (1)  $E(\mathcal{F})$  holds.
- (2)  $(\forall F : X \rightarrow c_0)(\exists Y \in \mathcal{F}) \alpha \circ F(Y) \in \mathcal{K}_\sigma$ .
- (3)  $(\forall \varphi : X \rightarrow {}^\omega\omega)(\exists Y \in \mathcal{F}) \varphi(Y) \in \mathcal{K}_\sigma$ .

*Proof.* (1)  $\Leftrightarrow$  (2) holds by Lemma 1.7 (a).

(2)  $\Rightarrow$  (3): If  $\varphi : X \rightarrow {}^\omega\omega$ , then  $\beta \circ \varphi : X \rightarrow c_0$  and so by (2) there is  $Y \in \mathcal{F}$  such that  $\theta \circ \varphi(Y) = \alpha \circ \beta \circ \varphi(Y) \in \mathcal{K}_\sigma$ . Hence  $\varphi(Y) \in \mathcal{K}_\sigma$  by Lemma 1.5.

The implication (3)  $\Rightarrow$  (2) is trivial because  $\alpha \circ F : X \rightarrow {}^\omega\omega$  in (2).  $\square$

We denote  $\mathcal{I}^+ = \mathcal{P}(X) \setminus \mathcal{I}$  for an ideal  $\mathcal{I} \subseteq \mathcal{P}(X)$ .

**Corollary 1.9.** *Let  $\mathcal{F} \subseteq \mathcal{P}(X)$ . Then  $E(\mathcal{F})$  holds if and only if  $E(\mathcal{I}^+)$  holds for every  $\sigma$ -ideal  $\mathcal{I} \subseteq \mathcal{P}(X) \setminus \mathcal{F}$ .  $\square$*

*Proof.* If  $E(\mathcal{F})$  does not hold, then by (3) in Theorem 1.8 there exists  $\varphi : X \rightarrow {}^\omega\omega$  such that  $\varphi(Y) \notin \mathcal{K}_\sigma$  for all  $Y \in \mathcal{F}$ . This means that  $\varphi^{-1}(\mathcal{K}_\sigma) \subseteq \mathcal{P}(X) \setminus \mathcal{F}$  is a  $\sigma$ -ideal on  $X$ , where  $\varphi^{-1}(\mathcal{K}_\sigma) = \{Y \subseteq X : \varphi(Y) \in \mathcal{K}_\sigma\}$ . Then  $\varphi$  witnesses that  $E((\varphi^{-1}(\mathcal{K}_\sigma))^+)$  does not hold. Conversely, if there exists an ideal  $\mathcal{I} \subseteq \mathcal{P}(X) \setminus \mathcal{F}$  such that  $E(\mathcal{I}^+)$  does not hold, then  $E(\mathcal{F})$  does not hold because  $\mathcal{F} \subseteq \mathcal{I}^+$ .  $\square$

**Corollary 1.10.** *If  $|X| \leq \mathfrak{c}$  and  $\mathcal{F} \subseteq \mathcal{P}(X)$ , then  $E(\mathcal{F})$  is equivalent to each of the following conditions:*

- (4)  $(\forall \varphi : X \xrightarrow{1-1} {}^\omega\omega)(\exists Y \in \mathcal{F}) \varphi(Y) \in \mathcal{K}_\sigma$ .
- (5)  $(\forall \varphi : X \xrightarrow{1-1} {}^\omega\omega)(\exists Z \in \mathcal{K}_\sigma) \varphi^{-1}(Z) \in \mathcal{F}$ .

*Proof.* Let us fix a one-to-one enumeration  $\{x_\alpha : \alpha < |X|\}$  of the set  $X$  and let us assume that (4) holds. We verify (3) in Theorem 1.8. Let  $\varphi : X \rightarrow {}^\omega\omega$  be arbitrary. By induction on  $\alpha < |X|$  we define  $\varphi'(x_\alpha) = \varphi(x_\alpha) + y_\alpha$  where  $y_\alpha \in {}^\omega 2$  is such that  $\varphi'(x_\alpha) \neq \varphi'(x_\beta)$  for all  $\beta < \alpha$ . So  $\varphi' : X \rightarrow {}^\omega\omega$  is one-to-one and  $\varphi(x) \leq \varphi'(x) \leq \varphi(x) + 1$ . By (4) there is  $Y \in \mathcal{F}$  such that  $\varphi'(Y) \in \mathcal{K}_\sigma$  and then also  $\varphi(Y) \in \mathcal{K}_\sigma$ .

Let  $\varphi : X \xrightarrow{1-1} {}^\omega\omega$ . If (4) holds and  $Y \in \mathcal{F}$  such that  $Z = \varphi(Y) \in \mathcal{K}_\sigma$ , then  $\varphi^{-1}(Z) = Y \in \mathcal{F}$  and so (5) holds. If (5) holds and  $Z \in \mathcal{K}_\sigma$  is such that  $Y = \varphi^{-1}(Z) \in \mathcal{F}$ , then  $\varphi(Y) = Z \in \mathcal{K}_\sigma$  and so (4) holds.  $\square$

**Corollary 1.11.** *If  $|X| = \mathfrak{c}$ ,  $\mathcal{F} \subseteq \mathcal{P}(X)$ , and there is  $A \subseteq X$  of size  $\mathfrak{c}$  such that  $Y \setminus A \in \mathcal{F}$  for all  $Y \in \mathcal{F}$ , then  $E(\mathcal{F})$  is equivalent to each of the following conditions:*

- (6)  $(\forall \varphi : X \rightarrow {}^\omega\omega \text{ bijective})(\exists Y \in \mathcal{F}) \varphi(Y) \in \mathcal{K}_\sigma.$
- (7)  $(\forall \varphi : {}^\omega\omega \rightarrow X \text{ bijective})(\exists Y \in \mathcal{F}) \varphi^{-1}(Y) \in \mathcal{K}_\sigma.$
- (8)  $(\forall \chi : {}^\omega\omega \rightarrow X \text{ surjective})(\exists Z \in \mathcal{K}_\sigma) \chi(Z) \in \mathcal{F}.$

*Proof.* (6) and (7) are equivalent. Obviously condition (4) from Corollary 1.10 implies (6). We prove that (6) implies (4). Let  $\varphi : X \xrightarrow{1-1} {}^\omega\omega$  be arbitrary. As  $|A| = \mathfrak{c}$ , we can find  $\psi : A \rightarrow {}^\omega\omega$  such that the mapping  $\varphi' = \varphi \upharpoonright (X \setminus A) \cup \psi$  is a bijection from  $X$  onto  ${}^\omega\omega$ . Applying (6) to  $\varphi'$  we find  $Y \in \mathcal{F}$  such that  $\varphi'(Y) \in \mathcal{K}_\sigma$ . Then  $Y \setminus A \in \mathcal{F}$  and  $\varphi(Y \setminus A) = \varphi'(Y \setminus A) \subseteq \varphi'(Y) \in \mathcal{K}_\sigma$ . Therefore  $E(\mathcal{F})$  is equivalent to (4).

Clearly (8) implies (6); we prove that (4) from Corollary 1.10 implies (8). If  $\chi : {}^\omega\omega \rightarrow X$  is an arbitrary surjective function, then let  $\varphi$  be its arbitrary right inverse function, i.e.,  $\chi \circ \varphi = \text{id}_X$ . As  $\varphi$  is injective there is  $Y \in \mathcal{F}$  such that  $Z = \varphi(Y)$  is in  $\mathcal{K}_\sigma$ . Then  $\chi(Z) = Y \in \mathcal{F}$ .  $\square$

Let us note that we cannot remove “surjective” from (8) because it would be equivalent to the assertion  $(\forall A \subseteq X) E(\mathcal{F} \upharpoonright A)$  (where  $\mathcal{F} \upharpoonright A = \{Y \in \mathcal{F} : Y \subseteq A\}$ ).

**Corollary 1.12.** *Let  $\mathcal{F} \subseteq \mathcal{P}(X)$ . If  $\mathcal{F}$  is closed under supersets, then  $E(\mathcal{F})$  is equivalent to the assertion:*

- (9)  $(\forall \varphi : X \rightarrow {}^\omega\omega)(\exists Z \in \mathcal{K}_\sigma) \varphi^{-1}(Z) \in \mathcal{F}.$

*Proof.* The implication from (9) to (3) does not require any assumptions because if  $Z \in \mathcal{K}_\sigma$  and  $Y = \varphi^{-1}(Z) \in \mathcal{F}$ , then  $\varphi(Y) = Z \in \mathcal{K}_\sigma$ . Conversely, if  $Y \in \mathcal{F}$  and  $Z = \varphi(Y) \in \mathcal{K}_\sigma$ , like in (3), then  $Y \subseteq \varphi^{-1}(Z)$ . Hence, if  $\mathcal{F}$  is closed under supersets, then  $\varphi^{-1}(Z) \in \mathcal{F}$ .  $\square$

Let  $\mathcal{F} \subseteq \mathcal{P}(X)$  and let  $\mathcal{I} = \mathcal{P}(X) \setminus \mathcal{F}$ . If  $\varphi : X \rightarrow {}^\omega\omega$  violates (9), then the pair of functions  $\varphi : X \rightarrow {}^\omega\omega$  and  $\psi = \varphi^{-1} : \mathcal{K}_\sigma \rightarrow \mathcal{I}$  form a Galois-Tukey embedding  $(X, \mathcal{I}, \in) \preceq ({}^\omega\omega, \mathcal{K}_\sigma, \in)$  in the notation of [1], because  $\varphi(x) \in Z$  implies  $x \in \psi(Z)$  for  $x \in X$  and  $Z \in \mathcal{K}_\sigma$ . Conversely, if a pair of functions  $(\varphi, \psi)$  is such an embedding, then  $\varphi$  violates (3) provided that  $\mathcal{F}$  is closed under supersets because if  $\varphi(Y) = Z \in \mathcal{K}_\sigma$ , then  $Y \subseteq \psi(Z) \in \mathcal{I}$ . A little modification of this proof gives an embedding  $(X, \mathcal{I}, \in) \preceq ({}^\omega\omega, {}^\omega\omega, \leq^*)$  because  $({}^\omega\omega, {}^\omega\omega, \leq^*) \simeq ({}^\omega\omega, \mathcal{K}_\sigma, \in)$ . This proves the following theorem.

**Theorem 1.13.** *Let  $\mathcal{F} \subseteq \mathcal{P}(X)$  be closed under supersets and let  $\mathcal{I} = \mathcal{P}(X) \setminus \mathcal{F}$ . Then the following conditions are equivalent:*

- (1)  $\neg E(\mathcal{F})$  holds.
- (2)  $(X, \mathcal{I}, \in) \preceq ({}^\omega\omega, \mathcal{K}_\sigma, \in).$
- (3)  $(X, \mathcal{I}, \in) \preceq ({}^\omega\omega, {}^\omega\omega, \leq^*).$   $\square$

Let us note that for the implications (1)  $\rightarrow$  (2)  $\rightarrow$  (3)  $\rightarrow$  (2) we do not need the assumption that  $\mathcal{F}$  is closed under supersets and the following cardinal inequalities are consequences of the embedding  $(X, \mathcal{I}, \in) \preceq ({}^\omega\omega, {}^\omega\omega, \leq^*).$

**Corollary 1.14.** *Let  $\mathcal{F} \subseteq \mathcal{P}(X)$  and let  $\mathcal{I} = \mathcal{P}(X) \setminus \mathcal{F}$ . Then  $\neg E(\mathcal{F})$  implies  $\mathfrak{b} \leq \text{non}(\mathcal{I})$  and  $\text{cov}(\mathcal{I}) \leq \mathfrak{d}$ .*  $\square$

**Corollary 1.15.** *Let  $\mathcal{I} = \mathcal{P}(X) \setminus \mathcal{F}$  and let  $L(\mathcal{F})$  denote the statement:*

$$(\exists Z \subseteq {}^\omega\omega, |Z| \geq |X|)(\forall Y \subseteq Z) Y \in \mathcal{K}_\sigma \text{ implies } |Y| < \text{non}(\mathcal{I}).$$

*If  $|X| \leq \mathfrak{c}$ , then  $L(\mathcal{F})$  implies  $\neg E(\mathcal{F})$ .*

*Proof.* An easy application of condition (4) from Corollary 1.10.  $\square$

Condition (3) of Theorem 1.8 has these easy consequences:  $\mathfrak{b}$  is the least cardinal  $\kappa$  such that there is a uniform ultrafilter  $\mathcal{F}$  on  $\kappa$  so that  $\neg E(\mathcal{F})$  holds;  $\mathfrak{b}$  is the least cardinal  $\kappa$  such that  $\neg E(\mathcal{F})$  holds for each uniform ultrafilter  $\mathcal{F}$  on  $\kappa$ .

We say that  $\kappa$  is an *E-cardinal*, if there exists a uniform ultrafilter  $\mathcal{F}$  on  $\kappa$  such that  $E(\mathcal{F})$  holds. By Corollary 1.14 each  $\kappa < \mathfrak{b}$  is an *E-cardinal* and every measurable cardinal  $\kappa$  is an *E-cardinal* because, if  $\mathcal{F}$  is a  $\kappa$ -complete ultrafilter on  $\kappa$ , then  $\text{cov}(\mathcal{P}(\kappa) \setminus \mathcal{F}) = \kappa > \mathfrak{d}$ .

*Question 1.16.* When a cardinal is an *E-cardinal*?

## 2. QUASINORMAL VERSUS UNIFORM

Now we try to compare the assertions  $E(\mathcal{F})$  and  $E^*(\mathcal{F})$ .

Shortly we will say that  $\mathcal{F} \subseteq \mathcal{P}(X)$  is *closed*, if  $\mathcal{F}$  is closed under supersets, i.e., if  $A \subseteq B \subseteq \mathcal{F}$  and  $A \in \mathcal{F}$ , then  $B \in \mathcal{F}$ . Let us consider some properties for  $\mathcal{F}$ . All next properties ensure that  $\mathcal{F}$  is closed. Without the closedness assumptions these properties would be more complex.

- (F0)  $\mathcal{F}$  is closed and if  $\bigcup_{m \in \omega} A_{n,m} \in \mathcal{F}$  for all  $n \in \omega$ , then there is  $f \in {}^\omega\omega$  such that  $\bigcup_{n \in \omega} \bigcup_{m \leq f(n)} A_{n,m} \in \mathcal{F}$ .
- (F1)  $\mathcal{F} = \bigcap_{n \in \omega} \mathcal{F}_n$  with  $\mathcal{F}_n \subseteq \mathcal{P}(X)$  closed such that whenever  $m \in \omega$  and  $A_n \notin \mathcal{F}_m$  for  $n \in \omega$ , then there is  $k$  such that  $\bigcup_{n \in \omega} A_n \notin \mathcal{F}_k$ .
- (F2)  $\mathcal{F} = \bigcap_{x \in [0,1]} \mathcal{F}_x$  with  $\mathcal{F}_x \subseteq \mathcal{P}(X)$  closed such that whenever  $A_n \subseteq X$  and  $A_n \notin \mathcal{F}_{x_n}$  for  $n \in \omega$ , then for every  $x$  with  $1 \geq x > \sup_n x_n$  there is  $y < x$  such that  $\bigcup_{n \in \omega} A_n \notin \mathcal{F}_y$ .
- (F3)  $\mathcal{F} = \bigcap_{p \in P} \mathcal{F}_p$  where  $(P, \leq)$  is a partially ordered set and  $\langle \mathcal{F}_p : p \in P \rangle$  is a system of closed subsets of  $\mathcal{P}(X)$  such that
  - (1) For all  $p, q \in P$ ,  $p \leq q$  implies  $\mathcal{F}_p \supseteq \mathcal{F}_q$ .
  - (2) If  $\bigcup_{n \in \omega} Y_n \in \mathcal{F}_p$  for all  $p \in P$ , then for every  $p \in P$  there is  $n \in \omega$  such that  $Y_n \in \mathcal{F}_p$ .

*Example 2.1.* (a) In [8] the case when  $\mathcal{F} = \{Y \subseteq X : \mu^*(Y) = \mu^*(X)\}$  for some finite upward continuous monotone outer measure  $\mu^*$  on  $X$  is considered. Then  $\mathcal{F}$  satisfies (F2) with  $\mathcal{F}_x = \{Y \subseteq X : \mu^*(Y)/\mu^*(X) \geq x\}$ . The corresponding system  $\mathcal{I}$  for Theorem 1.13 is the family of all sets  $A \subseteq X$  such that  $\mu^*(A) < \mu^*(X)$ . Vice versa, if  $\mathcal{F}_x$  for  $x \in [0, 1]$  are as in (F2), then  $\mu^*(Y) = \max\{x \in [0, 1] : Y \in \bigcap_{y < x} \mathcal{F}_y\}$  is an outer measure with the stated properties. Notice that for condition (F1) the formula  $\mu^*(Y) = \sup\{1 - 2^{-n} : Y \in \bigcap_{m < n} \mathcal{F}_m\}$  defines a finite monotone outer measure which is upward continuous (only) at the value 1.

(b) Let  $X$  be a topological space and let  $P$  be a  $\pi$ -base of open sets in  $X$ . Let  $\mathcal{F}_p$  for  $p \in P$  be the system of all sets  $A \subseteq X$  such that  $A \cap p$  is not meager. Then  $\mathcal{F} = \bigcap_{p \in P} \mathcal{F}_p$  has property (F3) and  $\mathcal{I}$  is the system of all sets  $A \subseteq X$  which are somewhere meager, i.e., there is  $p \in P$  such that  $p \cap A$  is meager.

(c) The previous example is connected with Baire category. The measurability is involved in a similar way. Let  $P$  be the system of all perfect subsets of  $\mathbb{R}$  with

positive measure and let  $\mathcal{F}_p$  be the system of all sets  $A \subseteq \mathbb{R}$  such that  $A \cap q \neq \emptyset$  for every  $q \in P$  with  $q \subseteq p$ . Then  $\mathcal{F} = \bigcap_{p \in P} \mathcal{F}_p$  has the property (F3) and  $\mathcal{I}$  is the system of all sets in  $\mathbb{R}$  which are disjoint from a set of positive measure.

It is easy to see that the implications (F2)  $\Rightarrow$  (F1)  $\Rightarrow$  (F0) hold. Clearly, (F3) generalizes (F2) and (F1).

**Lemma 2.2.** *Let  $\mathcal{F} = \bigcap_{p \in P} \mathcal{F}_p$  where  $(P, \leq)$  is a partially ordered set with a cofinal subset of size  $< \mathfrak{b}$  and  $\langle \mathcal{F}_p : p \in P \rangle$  is a system of closed subsets of  $\mathcal{P}(X)$  with property (F3). Then  $E(\mathcal{F})$  holds if and only if  $E^*(\mathcal{F}_p)$  holds for all  $p \in P$ .*

*Proof.* Let  $D \subseteq P$  be a cofinal subset of  $P$  of size  $< \mathfrak{b}$  and let  $F : X \rightarrow c_0$ . If for every  $p \in D$  there is  $A_p \in \mathcal{F}_p$  such that  $F \rightrightarrows 0$  on  $A_p$ , then  $Y = \bigcup_{p \in D} A_p$  belongs to  $\bigcap_{p \in D} \mathcal{F}_p = \bigcap_{p \in P} \mathcal{F}_p = \mathcal{F}$ . Then  $F \xrightarrow{\text{QN}} 0$  on  $Y$  since  $|D| < \mathfrak{b}$ . It follows that  $E(\mathcal{F})$  holds.

Conversely, if  $F : X \rightarrow c_0$ , then as we assume  $E(\mathcal{F})$ , there is  $Y \in \mathcal{F}$  such that  $F \xrightarrow{\text{QN}} 0$  on  $Y$ . Let  $Y_n \subseteq X$  for  $n \in \omega$  be such that  $Y = \bigcup_{n \in \omega} Y_n$  and  $F \rightrightarrows 0$  on  $Y_n$  for all  $n \in \omega$ . By condition (2), for every  $p \in P$ , there is  $n \in \omega$  such that  $Y_n \in \mathcal{F}_p$ . Consequently,  $E^*(\mathcal{F}_p)$  holds for all  $p \in P$ .  $\square$

**Lemma 2.3.** *Let  $\langle \mathcal{F}_p : p \in P \rangle$  be a system of closed subsets of  $\mathcal{P}(X)$  such that  $\mathcal{F} = \bigcap_{p \in P} \mathcal{F}_p = \bigcap_{p \in D} \mathcal{F}_p$  for some set  $D \subseteq P$  of size  $< \mathfrak{b}$ . If  $E^*(\mathcal{F}_p)$  holds for all  $p \in D$ , then  $E(\mathcal{F})$  holds.  $\square$*

### 3. THE CASE OF MEASURE AND CATEGORY

**Definition 3.1.** Let  $\mathcal{I} \subseteq \mathcal{P}(X)$  be an ideal and let  $\mathcal{B} \subseteq \mathcal{P}(X)$  be a set algebra.

- (1) We say that  $\mathcal{B}$  is a *covering* of  $\mathcal{P}(X)$  modulo  $\mathcal{I}$  if for every  $Y \subseteq X$  there exists  $B \in \mathcal{B}$  such that  $Y \subseteq B$  and  $(\forall B' \in \mathcal{B})(Y \subseteq B' \text{ implies } B \setminus B' \in \mathcal{I})$ .  $B$  is said to be a *cover* of  $Y$ .
- (2) We say that an ideal  $\mathcal{I}$  is  $\mathcal{B}$ -*homogeneous* if for every set  $B \in \mathcal{B}$  such that  $B \notin \mathcal{I}$  there exists a function  $f : X \rightarrow X$  such that  $f(X \setminus B) \subseteq B$  and  $Y \in \mathcal{I}$  if and only if  $f(Y) \in \mathcal{I}$  for all  $Y \subseteq X$ . (If we define  $f' : X \rightarrow B$  by  $f'(x) = x$  for  $x \in B$  and  $f'(x) = f(x)$  for  $x \in X \setminus B$ , then  $f'$  works, too.)
- (3) Let  $\mathcal{I}^+ = \mathcal{P}(X) \setminus \mathcal{I}$  and  $\mathcal{I}^{++} = \{A \subseteq X : (\forall B \in \mathcal{B} \setminus \mathcal{I}) B \cap A \notin \mathcal{I}\}$ .

It is well-known that the  $\sigma$ -algebra of Borel sets is a covering of  $\mathcal{P}(\mathbb{R})$  modulo the  $\sigma$ -ideal of meager sets  $\mathcal{M}$  as well as modulo the  $\sigma$ -ideal of null sets  $\mathcal{N}$ . The factor algebras  $\text{Borel}/\mathcal{M}$  and  $\text{Borel}/\mathcal{N}$  have both c.c.c..

**Lemma 3.2.** *Let  $\mathcal{I}_1$  and  $\mathcal{I}_2$  be ideals on  $X$  and let  $\mathcal{B} \subseteq \mathcal{P}(X)$  be a set algebra.*

- (1) *If  $\mathcal{B}$  is a covering of  $\mathcal{P}(X)$  modulo  $\mathcal{I}_1$  as well as modulo  $\mathcal{I}_2$ , then  $\mathcal{B}$  is a covering of  $\mathcal{P}(X)$  modulo  $\mathcal{I}_1 \cap \mathcal{I}_2$ .*
- (2) *If  $\mathcal{I}_1$  and  $\mathcal{I}_2$  are  $\mathcal{B}$ -homogeneous and  $\mathcal{B}$ -orthogonal (i.e., there exists  $A \in \mathcal{B}$  such that  $A \in \mathcal{I}_1$  and  $X \setminus A \in \mathcal{I}_2$ ), then  $\mathcal{I}_1 \cap \mathcal{I}_2$  is  $\mathcal{B}$ -homogeneous.*

*Proof.* (1) is trivial. We prove (2). Let us fix  $A \in \mathcal{B}$  such that  $A \in \mathcal{I}_1$  and  $X \setminus A \in \mathcal{I}_2$ . Let  $B \in \mathcal{B}$  be arbitrary such that  $B \notin \mathcal{I}_1 \cap \mathcal{I}_2$ . Then  $B \cap A \notin \mathcal{I}_2$  and  $B \setminus A \notin \mathcal{I}_1$  and so there are functions  $f_1 : X \rightarrow B \setminus A$  and  $f_2 : X \rightarrow B \cap A$  such that for every  $Y \subseteq X$ ,

$$Y \in \mathcal{I}_i \text{ if and only if } f_i(Y) \in \mathcal{I}_i, \quad i = 1, 2.$$

Let us define  $f : X \rightarrow B$  by  $f = (f_1 \upharpoonright (X \setminus A)) \cup (f_2 \upharpoonright A)$ .

If  $Y \in \mathcal{I}_1 \cap \mathcal{I}_2$ , then  $f(Y) = f_1(Y \setminus A) \cup f_2(Y \cap A)$ . Now  $f_1(Y \setminus A) \subseteq X \setminus A \in \mathcal{I}_2$  and  $f_2(Y \cap A) \subseteq A \in \mathcal{I}_1$ . Also  $f_1(Y \setminus A) \in \mathcal{I}_1$  and  $f_2(Y \cap A) \in \mathcal{I}_2$  because  $Y \setminus A \in \mathcal{I}_1$  and  $Y \cap A \in \mathcal{I}_2$ . Therefore  $f(Y) \in \mathcal{I}_1 \cap \mathcal{I}_2$ .

Conversely, if  $f(Y) \in \mathcal{I}_1 \cap \mathcal{I}_2$ , then  $f_1(Y \setminus A) \in \mathcal{I}_1$  and  $f_2(Y \cap A) \in \mathcal{I}_2$ . It follows that  $Y \setminus A \in \mathcal{I}_1$  and  $Y \cap A \in \mathcal{I}_2$  and consequently  $Y = (Y \cap A) \cup (Y \setminus A) \in \mathcal{I}_1 \cap \mathcal{I}_2$ .  $\square$

From now on let  $\mu$  denote the Lebesgue measure.

**Lemma 3.3.** *The ideals  $\mathcal{M}$ ,  $\mathcal{N}$ , and  $\mathcal{M} \cap \mathcal{N}$  are Borel-homogeneous ideals.*

*Proof.* (a) The case of  $\mathcal{M}$ . Let  $A_1 \cup A_2 = \mathbb{R}$  be a partition of  $\mathbb{R}$  into nonmeager Borel sets. There are meager sets  $M_i \subseteq A_i$  for  $i = 1, 2$  of size  $\mathfrak{c}$  such that  $A_1 \setminus M_1$  and  $A_2 \setminus M_2$  are homeomorphic to the Baire space  ${}^\omega\omega$ . Let  $g : A_1 \rightarrow A_2$  be any bijection such that the restriction  $g \upharpoonright (A_1 \setminus M_1) : A_1 \setminus M_1 \rightarrow A_2 \setminus M_2$  is a homeomorphism. Then  $f = g \cup g^{-1}$  is a bijective mapping on  $\mathbb{R}$  preserving the Baire category.

(a) The case of  $\mathcal{N}$ . We say that a set  $A \subseteq \mathbb{R}$  is *nowhere null*, if  $\mu(P \cap I) > 0$  for every interval  $I$  with  $A \cap I \neq \emptyset$ .

If  $P, Q \subseteq \mathbb{R}$  are perfect nowhere null nowhere dense sets of finite measure, then there is a homeomorphism  $h : P \rightarrow Q$  such that  $\mu(h(U))/\mu(Q) = \mu(U)/\mu(P)$  for every relatively open set  $U \subseteq P$ . To see this let us define  $f(x) = y$  if and only if  $\mu(P \cap (-\infty, x))/\mu(P) = \mu(Q \cap (-\infty, y))/\mu(Q)$ .

Now let  $A \cup B = \mathbb{R}$  be a partition of  $\mathbb{R}$  into measurable sets of positive measure. We can find infinite systems of disjoint perfect nowhere null nowhere dense sets of finite measure  $\{P_n : n \in \omega\}$  and  $\{Q_n : n \in \omega\}$  such that  $P_n \subseteq A$ ,  $Q_n \subseteq B$  for  $n \in \omega$ , and  $A' = A \setminus \bigcup_{n \in \omega} P_n$  and  $B' = B \setminus \bigcup_{n \in \omega} Q_n$  are null sets of size  $\mathfrak{c}$ . For  $n \in \omega$ , let  $f_n : P_n \rightarrow Q_n$  be the measure preserving homeomorphisms defined in the previous paragraph and  $g : A' \cup B' \rightarrow A' \cup B'$  be any bijection. Then the function  $f = g \cup g^{-1} \cup \bigcup_{n \in \omega} f_n \cup f_n^{-1}$  is a bijective mapping preserving null sets.

(c) Now the homogeneity of  $\mathcal{M} \cap \mathcal{N}$  follows by Lemma 3.2.  $\square$

For  $X \subseteq \mathbb{R}$  let  $\mathcal{N}(X)$  and  $\mathcal{M}(X)$  denote the ideal of measure zero subsets of  $X$  and the ideal of meager subsets  $X$ , respectively. We can ask about those  $X$  for which  $\mathcal{N}(X)$  or  $\mathcal{M}(X)$  is homogeneous with respect to relatively Borel subsets of  $X$ . Obviously, the proof of Theorem 3.4 works also if  $X$  is measurable or has the Baire property, respectively. On the other hand there may be (at least consistently with ZFC) also other such sets because if  $\mathcal{I}$  is an ideal on  $X$  such that  $\text{add}(\mathcal{I}) = \text{cof}(\mathcal{I})$ , then  $\mathcal{I}$  is  $\mathcal{P}(X)$ -homogeneous.

**Theorem 3.4.** *Let us assume that  $\mathcal{I} \subseteq \mathcal{P}(X)$  is a  $\sigma$ -ideal and  $\mathcal{B} \subseteq \mathcal{P}(X)$  is a  $\sigma$ -algebra which is a covering of  $\mathcal{P}(X)$  modulo  $\mathcal{I}$  such that  $\mathcal{B}/\mathcal{I}$  has c.c.c.. If  $\mathcal{I}$  is a  $\mathcal{B}$ -homogeneous ideal, then  $E(\mathcal{I}^+)$  implies  $E(\mathcal{I}^{++})$ , i.e., the following two conditions are equivalent:*

- (1)  $(\forall \varphi : X \rightarrow {}^\omega\omega)(\exists Y \in \mathcal{I}^+) \varphi(Y) \in \mathcal{K}_\sigma$ .
- (2)  $(\forall \varphi : X \rightarrow {}^\omega\omega)(\exists Y \in \mathcal{I}^{++}) \varphi(Y) \in \mathcal{K}_\sigma$ .

*Proof.* Let us assume that (1) holds and we prove (2). Let  $\varphi : X \rightarrow {}^\omega\omega$ .

We claim that for every  $B \in \mathcal{B} \setminus \mathcal{I}$  there exists  $Y \subseteq B$  such that  $Y \in \mathcal{I}^+$  and  $\varphi(Y) \in \mathcal{K}_\sigma$ . To see this let us fix  $B \in \mathcal{B} \setminus \mathcal{I}$ . Let  $f : X \rightarrow X$  be such that  $f(X \setminus B) \subseteq B$  and  $Y \in \mathcal{I}$  if and only if  $f(Y) \in \mathcal{I}$  for all  $Y \subseteq X$ . Let us define  $\varphi'(x) = \varphi(x)$  for  $x \in B$  and  $\varphi'(x) = \varphi(f(x))$  for  $x \in X \setminus B$ . By (1) there is  $Z \in \mathcal{I}^+$  such that  $\varphi'(Z) \in \mathcal{K}_\sigma$ . If  $Z \cap B \in \mathcal{I}^+$ , then we set  $Y = Z \cap B$ . Otherwise  $Z \setminus B \in \mathcal{I}^+$  and then we set  $Y = f(Z \setminus B)$ . In both cases  $Y \subseteq B$ ,  $Y \in \mathcal{I}^+$ , and  $\varphi(Y) \in \mathcal{K}_\sigma$ .

Now let  $F$  be a maximal system of pairs  $(Z, B)$  where  $Z \in \mathcal{I}^+$ ,  $\varphi(Z) \in \mathcal{K}_\sigma$ ,  $B$  is a cover of  $Z$  in  $\mathcal{B}$  modulo  $\mathcal{I}$ , and  $B \cap B' \in \mathcal{I}$  for distinct  $(Z, B), (Z', B') \in F$ . Let  $Y = \bigcup_{(Z, B) \in F} Z$ . Then  $Y \in \mathcal{I}^{++}$  and, as  $\mathcal{B}/\mathcal{I}$  has c.c.c.,  $F$  is countable and so  $\varphi(Y) \in \mathcal{K}_\sigma$ .  $\square$

Obviously the ideals  $\mathcal{M}$ ,  $\mathcal{N}$ , and  $\mathcal{M} \cap \mathcal{N}$  satisfy the assumptions of Theorem 3.4. Now we can say a bit more than Theorem 1.13 says for the assertion  $E(\mathcal{N}^{++})$ .

**Corollary 3.5.** (a) *The following conditions are equivalent:*

- (1)  $\neg E(\mathcal{N}^+)$ .
- (2)  $\neg E(\mathcal{N}^{++})$ .
- (3)  $([0, 1], \mathcal{N}, \in) \preceq (\omega_\omega, \omega_\omega, \leq^*)$ .
- (4)  $([0, 1], \{A \subseteq [0, 1] : \mu^*(A) < 1\}, \in) \preceq (\omega_\omega, \omega_\omega, \leq^*)$ .

(b) *The following conditions hold:*

- (1)  $\neg E(\mathcal{M}^+)$ .
- (2)  $\neg E(\mathcal{M}^{++})$ .
- (3)  $([0, 1], \mathcal{M}, \in) \preceq (\omega_\omega, \omega_\omega, \leq^*)$ .
- (4)  $([0, 1], \{A \subseteq [0, 1] : A \cap U \in \mathcal{M} \text{ for some open } U \neq \emptyset\}, \in) \preceq (\omega_\omega, \omega_\omega, \leq^*)$ .

(c) *The following conditions are equivalent:*

- (1)  $\neg E((\mathcal{M} \cap \mathcal{N})^+)$ .
- (2)  $\neg E((\mathcal{M} \cap \mathcal{N})^{++})$ .
- (3)  $([0, 1], \mathcal{M} \cap \mathcal{N}, \in) \preceq (\omega_\omega, \omega_\omega, \leq^*)$ .
- (4)  $([0, 1], \{A \subseteq [0, 1] : A \cap U \in \mathcal{M} \cap \mathcal{N} \text{ for some open } U \neq \emptyset\}, \in) \preceq (\omega_\omega, \omega_\omega, \leq^*)$ .

*Proof.* The equivalences hold by Theorem 3.4 and by Theorem 1.13 in all three cases. It is well-known that there exists a Galois-Tukey morphism  $([0, 1], \mathcal{M}, \in) \preceq (\omega_\omega, \omega_\omega, \leq^*)$  (see [1], [6]).  $\square$

**Corollary 3.6.**  $E(\mathcal{N}^+)$  holds if and only if  $E((\mathcal{M} \cap \mathcal{N})^+)$  holds.

*Proof.* Let us assume that  $E((\mathcal{M} \cap \mathcal{N})^+)$  holds and let  $F : \mathbb{R} \rightarrow c_0$  be given. As  $\neg E(\mathcal{M}^+)$  holds, there is  $H : \mathbb{R} \rightarrow c_0$  such that for every  $Y \subseteq \mathbb{R}$ , if  $H \xrightarrow{\text{QN}} 0$  on  $Y$ , then  $Y \in \mathcal{M}$ . Let  $G = \max\{F, H\}$ . Then there is  $Y \in (\mathcal{M} \cap \mathcal{N})^+$  such that  $G \xrightarrow{\text{QN}} 0$  on  $Y$ . As  $F \leq G$  also  $F \xrightarrow{\text{QN}} 0$  on  $Y$ . Then  $Y \in \mathcal{M}$  by the choice of  $H$  and so  $Y \in \mathcal{N}^+$ . It follows that  $E(\mathcal{N}^+)$  holds. The inverse implication is a consequence of the inclusion  $\mathcal{N}^+ \subseteq (\mathcal{M} \cap \mathcal{N})^+$ .  $\square$

*Question 3.7.* If  $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{P}(X)$ , then  $E(\mathcal{F}_1)$  implies  $E(\mathcal{F}_2)$ . This fact can be expressed by a Galois-Tukey embedding  $(X, \mathcal{P}(X) \setminus \mathcal{F}_1, \in) \preceq (X, \mathcal{P}(X) \setminus \mathcal{F}_2, \in)$  given by the pair of identity functions. Under what conditions an inverse embedding exists?

*Question 3.8.* Which implications of the form  $E(\mathcal{F}_1) \Rightarrow E(\mathcal{F}_2)$  with  $\mathcal{F}_1, \mathcal{F}_2 \subseteq \mathcal{P}(X)$  imply a Galois-Tukey embedding  $(X, \mathcal{P}(X) \setminus \mathcal{F}_2, \in) \preceq (X, \mathcal{P}(X) \setminus \mathcal{F}_1, \in)$ ?

The assertion  $E(\mathcal{N}^{++})$  is denoted by (GES) in [8]. By Corollary 3.5,  $\neg(\text{GES})$  is equivalent to  $\neg E(\mathcal{N}^+)$  which by Corollary 1.11 is equivalent to the condition

$$(\exists \varphi : [0, 1] \rightarrow \omega_\omega \text{ bijective})(\forall Y \in \mathcal{N}^+) \varphi(Y) \notin \mathcal{K}_\sigma.$$

Condition  $L(\mathcal{N}^+)$  from Corollary 1.15 states the existence of a non( $\mathcal{N}$ )- $\mathcal{K}_\sigma$ -Luzin set of cardinality  $\mathfrak{c}$  (see Definition 4.1 below). Let us consider another condition:

- (M) There exists a non-atomic real-valued  $\sigma$ -additive measure  $\nu$  on a  $\sigma$ -algebra of subsets of  ${}^\omega\omega$  such that  $\nu({}^\omega\omega) = 1$  and  $\nu(K) = 0$  for all  $K \in \mathcal{K}_\sigma$ .

Now it is obvious, that  $L(\mathcal{N}^+)$  implies  $\neg(\text{GES})$  and  $\neg(\text{GES})$  implies (M).

#### 4. OTHER QUESTIONS RELATED TO $E(\mathcal{F})$

##### Definition 4.1.

- (1) A sequence  $\langle f_\xi : \xi < \kappa \rangle$  in  ${}^\omega\omega$  is a *B-sequence* if for every  $f \in {}^\omega\omega$  there is  $\eta < \kappa$  such that  $f_\xi \not\leq^* f$  for all  $\xi > \eta$ . A cardinal  $\kappa$  is a *B-cardinal* (i.e.,  *$\mathfrak{b}$  like cardinal number*), if there exists a *B-sequence* of length  $\kappa$ .
- (2) Let  $\mathcal{I} \subseteq \mathcal{P}(X)$ . A set  $Y \subseteq X$  is a  *$\kappa$ - $\mathcal{I}$ -Luzin set* if  $|Y| \geq \kappa$  and  $|Y \cap A| < \kappa$  for all  $A \in \mathcal{I}$ .
- (3) Let  $F \subseteq \mathcal{P}(\omega)$  be a filter. For  $f, g \in {}^\omega\omega$  we define

$$f \leq_F g \equiv (\exists A \in F)(\forall n \in A) f(n) \leq g(n).$$

Let  $\mathfrak{b}_F$  and  $\mathfrak{d}_F$  denote, respectively, the unbounding number and the dominating number for this quasi-ordering of  ${}^\omega\omega$ .

Let us note that  $\mathfrak{b} = \mathfrak{b}_F$  and  $\mathfrak{d} = \mathfrak{d}_F$  for Fréchet filter  $F$ . If  $F \subseteq F'$  are filters that extend Fréchet filter, then  $\mathfrak{b}_F \leq \mathfrak{b}_{F'} \leq \mathfrak{d}_{F'} \leq \mathfrak{d}_F$ . The reader can find more information on cardinals  $\mathfrak{b}_F$  and  $\mathfrak{d}_F$  in [3].

Here we list several facts on *B*-cardinals and  $\mathcal{K}_\sigma$ -Luzin sets:

##### Theorem 4.2.

- (1)  $\mathfrak{b}$  is the least *B-cardinal* and if a filter  $F$  contains Fréchet filter, then  $\mathfrak{b}_F$  and  $\mathfrak{d}_F$  are *B-cardinals*.
- (2)  $\kappa$  is a *B-cardinal* if and only if  $\text{cf } \kappa$  is a *B-cardinal*.
- (3) If  $\kappa$  is a regular *B-cardinal*, then  $\mathfrak{b} \leq \kappa \leq \mathfrak{d}$ .
- (4) If  $\kappa \leq \mathfrak{c}$  is a *B-cardinal*, then there exists a  $\kappa$ - $\mathcal{K}_\sigma$ -Luzin set  $X \subseteq {}^\omega\omega$  such that  $\kappa$  is the least cardinal such that  $X$  is a  $\kappa$ - $\mathcal{K}_\sigma$ -Luzin set.
- (5) If there exists a  $\kappa$ - $\mathcal{K}_\sigma$ -Luzin set  $X \subseteq {}^\omega\omega$  of size  $\lambda$ , then  $\mathfrak{b} \leq \kappa$  and every regular cardinal  $\mu$  with  $\kappa \leq \mu \leq \lambda$  is a *B-cardinal*. If moreover  $\kappa < \lambda$ , then  $\lambda \leq \mathfrak{d}$ .
- (6) A cardinal  $\kappa$  is a *B-cardinal* if and only if there exists a  $\text{cf}(\kappa)$ - $\mathcal{K}_\sigma$ -Luzin set in  ${}^\omega\omega$ .
- (7) If there exists a  $\kappa$ -Luzin set  $X$  of size  $\lambda$ , then  $\text{non}(\mathcal{M}) \leq \kappa$  and  $\mu \leq \text{cov}(\mathcal{M})$  for every regular cardinal  $\mu$  with  $\kappa \leq \mu \leq \lambda$ . If moreover  $\kappa < \lambda$ , then  $\mathfrak{b} = \text{add}(\mathcal{M}) \leq \text{non}(\mathcal{M}) \leq \kappa < \lambda \leq \text{cov}(\mathcal{M}) \leq \text{cof}(\mathcal{M}) = \mathfrak{d}$ .
- (8) If  $\text{cov}(\mathcal{M}) = \text{cof}(\mathcal{M}) = \kappa$ , then there exists a  $\kappa$ -Luzin set of size  $\kappa$ .

*Proof.* (1) Let  $\langle f_\xi : \xi < \mathfrak{b} \rangle$  be an  $\leq_F$ -unbounded system of functions such that  $f_\xi \leq_F f_\eta$  for  $\xi < \eta < \mathfrak{b}$  and let  $\langle g_\xi : \xi < \mathfrak{d} \rangle$  be a  $\leq_F$ -dominating system of functions such that  $g_\xi \not\leq_F g_\eta$  for  $\xi < \eta < \mathfrak{d}$ . Both these sequences are *B-sequences*.

For (2) it is enough to realize that elements in a *B-sequence* can repeat (not cofinally many times) and every cofinal subsequence of a *B-sequence* is a *B-sequence*.

(3) If  $\kappa$  is a regular *B-cardinal*, then  $\mathfrak{b} \leq \kappa$  by (1). We prove that also  $\kappa \leq \mathfrak{d}$ ; by (2),  $\text{cf } \mathfrak{d}$  is a regular *B-cardinal*. Let  $\langle f_\xi : \xi < \kappa \rangle$  be any *B-sequence* and let  $D \subseteq {}^\omega\omega$  be a dominating family of size  $\mathfrak{d}$ . For  $f \in D$  let  $X_f = \{\xi < \kappa : f_\xi \leq^* f\}$ . Clearly  $|X_f| < \kappa$  and  $\kappa = \bigcup_{f \in D} X_f$ . It follows that  $|D| \geq \kappa$  because  $\kappa$  is regular.

(4) Let  $\langle f_\xi : \xi < \text{cf } \kappa \rangle$  be a  $B$ -sequence and let  $D_\xi \subseteq {}^\omega 2$  for  $\xi < \text{cf } \kappa$  be such that  $|D_\xi| < \kappa$  and  $|\bigcup_{\xi < \text{cf } \kappa} D_\xi| = \kappa$ . Define  $X = \{f + g : (\exists \xi < \text{cf } \kappa) f = f_\xi \text{ and } g \in D_\xi\}$ . Clearly  $|X| = \kappa$  and  $X$  is a  $\kappa$ - $\mathcal{K}_\sigma$ -Luzin set with  $\kappa$  minimal.

(5) If  $\sigma = \langle f_\xi : \xi < \mu \rangle$  is a sequence of distinct elements of  $X$ , then  $\sigma$  is a  $B$ -sequence whenever  $\text{cf } \mu \geq \kappa$ . The inequality  $\mathfrak{b} \leq \kappa$  holds because every subset of  $X$  of size  $< \mathfrak{b}$  is bounded, and hence has size  $< \kappa$ . By an argument similar to the proof of (3) we can see that if  $\kappa < \lambda$  then  $\lambda \leq \kappa \mathfrak{d} = \mathfrak{d}$ .

(6) By (2) we can restrict to regular cardinals. For regular cardinals the assertion follows by (4) and (5).

(7)  $\text{non}(\mathcal{M}) \leq \kappa$  because every subset of  $X$  of size  $< \text{non}(\mathcal{M})$  is meager and hence has size  $< \kappa$ . Let us assume that  $\kappa \leq \mu \leq \lambda$  and  $\mu$  is regular. If  $C$  is a system of meager subsets of  $X$  covering  $X$ , then  $|\bigcup C| \geq \mu$  and as each set in  $C$  has size  $< \mu$  and  $\mu$  is regular, it follows that  $|C| \geq \mu$ . Therefore  $\text{cov}(\mathcal{M}) \geq \mu$ . So, if  $\kappa < \lambda$ , then  $\text{cov}(\mathcal{M}) \geq \lambda$ , and the rest is a consequence of the equalities  $\text{add}(\mathcal{M}) = \min\{\text{cov}(\mathcal{M}), \mathfrak{b}\}$  and  $\text{cof}(\mathcal{M}) = \max\{\text{non}(\mathcal{M}), \mathfrak{d}\}$  (see [6], [2]).

(8) Let  $\langle M_\alpha : \alpha < \kappa \rangle$  be an enumeration of a base of  $\mathcal{M}$ . By induction choose  $x_\alpha \in \mathbb{R} \setminus \bigcup_{\beta < \alpha} M_\beta$ . Then  $X = \{x_\alpha : \alpha < \kappa\}$  is a  $\kappa$ -Luzin set.  $\square$

Let us note that  $\kappa$  is a  $B$ -cardinal if and only if there exists a Galois-Tukey morphism  $(\kappa, \kappa, \leq) \preceq ({}^\omega \omega, {}^\omega \omega, \leq^*)$ . For example, if  $\langle f_\xi : \xi < \kappa \rangle$  is a  $B$ -sequence then the pair of functions  $\varphi : \kappa \rightarrow {}^\omega \omega$  and  $\psi : {}^\omega \omega \rightarrow \kappa$  defined by  $\varphi(\xi) = f_\xi$  and  $\psi(f) = \sup\{\xi : f_\xi \leq^* f\}$  is a morphism because  $\varphi(\xi) \leq^* f$  implies  $\xi \leq \psi(f)$ . This fact gives another argument for conditions (2) and (3). We do not know whether supremum of regular  $B$ -cardinals can be strictly smaller than  $\mathfrak{d}$  (in this case, by (1),  $\mathfrak{d}$  must be singular).

By adding Cohen reals we obtain a model in which  $\mathfrak{b} = \omega_1 < \mathfrak{c}$  and there is an  $\omega_1$ - $\mathcal{K}_\sigma$ -Luzin set of cardinality  $\mathfrak{c}$ . In this model every regular cardinal with  $\omega_1 \leq \kappa \leq \mathfrak{c}$  is the cofinality  $\mathfrak{d}_U$  of an ultraproduct  ${}^\omega \omega / U$  for some ultrafilter  $U$  on  $\omega$  (see [5]).

Using  $B$ -cardinals we can rewrite Corollary 1.15 as follows (compare with [8, Proposition 8]):

**Corollary 4.3.** *Let  $|X| \leq \mathfrak{c}$ , let  $\mathcal{F} \subseteq \mathcal{P}(X)$ , and let  $\mathcal{I} = \mathcal{P}(X) \setminus \mathcal{F}$ .*

- (1) *If there exists a  $\text{non}(\mathcal{I})$ - $\mathcal{K}_\sigma$ -Luzin set of size  $|X|$ , then  $\neg E(\mathcal{F})$  holds.*
- (2) *If  $|X|$  is a  $B$ -cardinal and  $\text{non}(\mathcal{I}) = |X|$ , then  $\neg E(\mathcal{F})$  holds.*  $\square$

Notice that (2) is a special case of (1).

In the case of measure Corollary 4.3 states:

**Corollary 4.4** ([8, Proposition 8]).

- (1) *If there exists a  $\text{non}(\mathcal{N})$ - $\mathcal{K}_\sigma$ -Luzin set of size  $\mathfrak{c}$ , then  $\neg(\text{GES})$  holds.*
- (2) *If  $\mathfrak{c}$  is a  $B$ -cardinal and  $\text{non}(\mathcal{N}) = \mathfrak{c}$ , then  $\neg(\text{GES})$  holds.*  $\square$

Let us recall a theorem of W. Sierpiński ([9, Proposition  $P_3$ ]): CH holds if and only if there are functions  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  for  $n \in \omega$  such that  $(\forall A \in [\mathbb{R}]^{\geq \omega_1})(\forall^\infty n \in \omega) f_n(A) = \mathbb{R}$ . This theorem was a motivation for the next characterization of  $\kappa$ - $\mathcal{K}_\sigma$ -Luzin sets.

**Theorem 4.5.** *The following conditions are equivalent for any  $\kappa \leq \lambda \leq \mathfrak{c}$  with  $\text{cf } \kappa \geq \omega_1$ :*

- (1) *There is a  $\kappa$ - $\mathcal{K}_\sigma$ -Luzin set of size  $\lambda$ .*

- (2) *There are functions  $f_n : \lambda \rightarrow \omega$  for  $n \in \omega$  such that  $(\forall A \in [\lambda]^\kappa)(\forall^\infty n \in \omega)$   
 $|f_n(A)| = \omega$ .*
- (3) *There are functions  $f_n : \lambda \rightarrow \omega$  for  $n \in \omega$  such that  $(\forall A \in [\lambda]^\kappa)(\exists n \in \omega)$   
 $|f_n(A)| = \omega$ .*

*Proof.* (1)  $\Rightarrow$  (3): Let  $L \subseteq {}^\omega\omega$  be a  $\kappa$ - $\mathcal{K}_\sigma$ -Luzin set of size  $\lambda$  and let  $\{g_\xi : \xi < \lambda\}$  be a one-to-one enumeration of  $L$ . Let us define  $f_n : \lambda \rightarrow \omega$  by  $f_n(\xi) = g_\xi(n)$ . To obtain a contradiction let us assume that there is  $A \in [\lambda]^\kappa$  such that  $|f_n(A)| < \omega$  for all  $n \in \omega$ . Hence there is  $h \in {}^\omega\omega$  such that  $f_n(\xi) \leq h(n)$  for all  $n$ . Then  $\{g_\xi : \xi \in A\} \in \mathcal{K}_\sigma$  which is a contradiction.

(3)  $\Rightarrow$  (1): Let  $f_n : \lambda \rightarrow \omega$  for  $n \in \omega$  satisfy (3). Let us define  $g_\xi \in {}^\omega\omega$  by  $g_\xi(n) = f_n(\xi)$ . By induction on  $\xi < \lambda$  let us define  $h_\xi = g_\xi + y_\xi$  where  $y_\xi \in {}^\omega 2$  is such that  $h_\xi \neq h_\eta$  for all  $\eta < \xi$ . Then the set  $L = \{h_\xi : \xi < \lambda\}$  has size  $\lambda$  and we prove that it is a  $\kappa$ - $\mathcal{K}_\sigma$ -Luzin set. To obtain a contradiction let us assume that we have  $A \in [\lambda]^\kappa$  such that  $\{h_\xi : \xi \in A\} \in \mathcal{K}_\sigma$ . As cf  $\kappa > \omega$ , there is  $B \in [A]^\kappa$  and  $h \in {}^\omega\omega$  such that  $h_\xi \leq h$  for all  $\xi \in B$ . Then  $f_n(\xi) \leq h_\xi(n) \leq h(n)$  for all  $\xi \in B$  and  $n \in \omega$  which contradicts condition (3).

(3)  $\Rightarrow$  (2): If  $f_n : \lambda \rightarrow \omega$  for  $n \in \omega$  satisfy (3), then  $f'_n(\xi) = \max\{f_i(\xi) : i \leq n\}$  for  $n \in \omega$  satisfy (2). The implication (2)  $\Rightarrow$  (3) is trivial.  $\square$

For a while let us consider a special case of Corollary 4.3:

*Let  $\mathcal{F} \subseteq \mathcal{P}(\mathbb{R})$  and  $\mathcal{I} = \mathcal{P}(\mathbb{R}) \setminus \mathcal{F}$  be such that  $\mathcal{F} \cap [\mathbb{R}]^{\leq \omega} = \emptyset$ . Then CH implies  $\neg E(\mathcal{F})$ . (Because  $\text{non}(\mathcal{I}) = \mathfrak{c}$  and there exists a  $\mathfrak{c}$ - $\mathcal{K}_\sigma$ -Luzin set of size  $\mathfrak{c}$ .)*

This special case has this application: If  $\mathcal{F} \subseteq \mathcal{P}(X)$  does not contain countable sets and the definition of  $\mathcal{F}$  does not contradict neither CH nor  $|X| = \omega_1$ , then  $E(\mathcal{F})$  is not provable in ZFC, i.e.,  $E(\mathcal{F})$  is independent from ZFC if and only if  $E(\mathcal{F})$  is consistent with ZFC.

*Examples.* 1. Let  $\mathcal{E} \subseteq \mathcal{P}(\mathbb{R})$  be the  $\sigma$ -ideal generated by closed sets of measure 0. Then  $\mathcal{E} \subseteq \mathcal{M} \cap \mathcal{N}$  and  $E((\mathcal{M} \cap \mathcal{N})^+)$  is consistent with ZFC by Corollary 3.6 (because  $E(\mathcal{N}^{++})$  is consistent, see [8] or [10]). As  $(\mathcal{M} \cap \mathcal{N})^+ \subseteq \mathcal{E}^+$ ,  $E(\mathcal{E}^+)$  is consistent with ZFC and hence independent from ZFC.

2. Let  $s^0$  be the Marczewski ideal. It is well-known that  $\mathfrak{d} < \text{cov}(s^0)$  holds in the forcing extension of a model of ZFC + CH via a countable support iteration of Sacks forcing of length  $\omega_2$  (see [1] and [7]). Therefore  $E((s^0)^+)$  is consistent by Corollary 1.14 and consequently  $E((s^0)^+)$  is independent from ZFC.

3. Let  $(s^0)^{++}$  be the family of all sets  $Y \subseteq \mathbb{R}$  such that  $Y \cap P \neq \emptyset$  for all perfect subsets  $P \subseteq \mathbb{R}$ . We prove  $\neg E((s^0)^{++})$ : Let  $\varphi : \mathbb{R} \rightarrow {}^\omega\omega$  be such that the restriction  $\varphi \upharpoonright \mathbb{I}r : \mathbb{I}r \rightarrow {}^\omega\omega$  is a homeomorphism from the set of irrational numbers onto the Baire space. Now, if there is  $f \in {}^\omega\omega$  such that  $\varphi(y) \leq^* f$  for all  $y \in Y$ , i.e.,  $\varphi(Y) \in \mathcal{K}_\sigma$ , then the set  $Z = \{x \in {}^\omega\omega : (\forall n \in \omega)(x(n) = f(n) + 1 \text{ or } x(n) = f(n) + 2)\}$  is compact perfect subset of  ${}^\omega\omega$  and  $\varphi^{-1}(Z) \cap \mathbb{I}r$  is a perfect set disjoint from  $Y$ . Therefore  $Y \notin (s^0)^{++}$ . It follows that  $\neg E((s^0)^{++})$  holds.

## REFERENCES

- [1] T. Bartoszyński, *Invariants of measure and category*, Handbook of Set-Theory (Foreman, Kanamori, and Magidor, eds.), preprint.
- [2] T. Bartoszyński and H. Judah, *Set Theory. On the Structure of the Real Line*. A. K. Peters, Ltd., Wellesley, MA, 1995. MR 96k:03002.

- [3] A. Blass and H. Mildenberger, *On the cofinality of ultrapowers*, J. Symbolic Logic **64** (1999), no. 2, 727–736. MR 2001h:03079.
- [4] L. Bukovský, I. Reclaw, and M. Repický, *Spaces not distinguishing pointwise and quasinormal convergence of real functions*, Topology Appl. **41** (1991), no. 1-2, 25–40. MR 93b:54037.
- [5] R. M. Canjar, *Countable ultraproducts without CH*, Ann. Pure Appl. Logic **37** (1988), no. 1, 1–79. MR 89g:03073.
- [6] D. Fremlin, *Cichoń's diagram*, Publ. Math. Univ. Pierre Marie Curie **66**, Semin. Initiation Anal. 23ème Année–1983/84, Exp. No. 5, (1984), pp. 1–13.
- [7] H. Judah, A. W. Miller, and S. Shelah, *Sacks forcing, Laver forcing, and Martin's axiom*. Arch. Math. Logic **31** (1992), no. 3, 145–161. MR 93e:03074.
- [8] R. Pinciroli, *On the independence of a generalized statement of Egoroff's theorem from ZFC after T. Weiss*, Real Anal. Exchange **32** (2006/07), no. 1, 225–232. MR 2008f:03068.
- [9] W. Sierpiński, *Hypothèse du continu*, Monografie Matematyczne, vol. 4, Warszawa-Lwów, 1934.
- [10] T. Weiss, *A note on generalized Egorov's theorem*, preprint 2004.

MATEMATICKÝ ÚSTAV SAV, JESENNÁ 5, 041 54 KOŠICE, SLOVAKIA  
DEPARTMENT OF COMPUTER SCIENCE, FACULTY OF SCIENCE, P. J. ŠAFÁRIK UNIVERSITY, JE-  
SENNÁ 5, 041 54 KOŠICE, SLOVAKIA

*E-mail address:* repicky@kosice.upjs.sk

*URL:* kosice.upjs.sk/~repicky