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Spaces not distinguishing convergences of real-valued functions

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Abstract

In [Topology Appl. 41 (1991) 25] we have introduced the notion of a wQN-space as a space in which for every sequence of continuous functions pointwisely converging to 0 there is a subsequence quasi-normally converging to 0. In the present paper we continue this investigation and generalize some concepts touched there. The content is a variety of notions and relationships among them. The result is another scale in the investigation of smallness and the question is how this scale fits with other known scales and whether all relations in it are proper.

Keywords: mQN-space; Σ-space; wQN-space; QN-space; $\overline{\text{QN}}$-space; ΣQN-space; wΣQN-space; Weak distributivity of a family of sets

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0. Introduction

In [6] we have introduced the notion of a wQN-space as a space in which for every sequence of continuous functions pointwisely converging to 0 there is a subsequence quasi-normally converging to 0, i.e., the wQN-space does not distinguish pointwise and quasi-normal convergence. The motivation for introducing this notion comes from the theory of thin sets of trigonometric series and actually our knowledge about wQN-spaces was used in [5]. The idea to study spaces which do not distinguish two kinds of convergence is not new. Iséki's characterizations of pseudo-compactness (see [16,4]) and of countable

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compactness [15] in terms of quasi-uniform or simply-uniform convergence can serve as examples. The referee has informed us about two other sources which show the importance of such study: [19], where Kliś considers normed spaces in which every sequence converging to 0 has a summable subsequence; and a Mathias' theorem [30, Theorem 31] which characterizes closed subsets of separable Banach spaces as those analytic subsets with the property that each sequence converging to 0 has a summable subsequence. Sometimes related notions were studied from another point of view. Fremlin [13] has investigated a notion of an s_1-space. Scheepers [26] has introduced an $S_1(\Gamma, \Gamma)$ property of a space. Then in [28], Scheepers shows that every space with $S_1(\Gamma, \Gamma)$ property is an s_1-space and every s_1-space is a wQN-space. Moreover, a Lindelöf wQN-space is an s_1-space.

The paper is organized as follows. In Section 1 we introduce in a rather systematic way several definitions of spaces not distinguishing some types of convergence and we present basic relations among them (the considered class of functions $\mathcal{F}$ is arbitrary). Section 2 contains some applications of previous results to the class $\mathcal{F}$ of Borel measurable functions. Sections 3 and 4 are devoted to the study of mQN-spaces. The main results of Section 5 are presented in Theorem 5.10. The title of Section 6 indicates its content. Here we investigate the properties of spaces related to Σ- and Σ^*-convergences. Diagram 2 summarizes results of previous sections. In Section 7 we present some examples of spaces with investigated properties.

We use the standard terminology and notation. For topological terminology see, e.g., [11]. Let us recall that a sequence of real-valued functions $\{f_n\}_{n=0}^{\infty}$ quasi-normally converges to a function f on a set X if for some sequence of positive reals $\{\varepsilon_n\}_{n=0}^{\infty}$ with $\lim_{n\to\infty} \varepsilon_n = 0$ we have $(\forall x)(\forall^{\infty} n)\ |f_n(x) - f(x)| \leqslant \varepsilon_n$. Replacing positive reals ε_n by $\varepsilon_n = 0$ we obtain the discrete convergence (see [3,7]). We generalize a notion considered by Denjoy [9]. We say that a sequence of real-valued functions $\{f_n\}_{n=0}^{\infty}$ pseudo-normally converges to a function f on a set X if for some sequence of positive reals $\{\varepsilon_n\}_{n=0}^{\infty}$ with $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ we have $(\forall x)(\forall^{\infty} n)\ |f_n(x) - f(x)| \leqslant \varepsilon_n$. Denjoy considers the case $f = 0$ and speaks about pseudo-normal convergence of the series $\sum_{n=0}^{\infty} f_n$. We shall need the following standard families of real-valued functions defined on a topological space X:

$$\begin{aligned}
\mathcal{M}(X) &= \{f \in {}^X\mathbb{R}\colon\ f \text{ is Borel measurable}\},\\
\mathcal{M}_1(X) &= \{f \in {}^X\mathbb{R}\colon\ f \text{ is } F_\sigma \text{ measurable}\},\\
\mathcal{A}m_1(X) &= \{f \in {}^X\mathbb{R}\colon\ f \text{ is } F_\sigma \cap G_\delta \text{ measurable}\},\\
\mathcal{B}_1(X) &= \{f \in {}^X\mathbb{R}\colon\ f \text{ is a pointwise limit of continuous functions}\},\\
\mathcal{D}_1(X) &= \{f \in {}^X\mathbb{R}\colon\ f \text{ is a discrete limit of continuous functions}\}.
\end{aligned}$$

Clearly $\mathcal{D}_1(X) \subseteq \mathcal{A}m_1(X) \subseteq \mathcal{M}_1(X)$ and $\mathcal{M}_1(X) = \mathcal{B}_1(X)$ whenever X is perfectly normal. Let us remark that $\mathcal{D}_1(X)$ consists of quasi-normal limits of sequences of continuous functions. Let us note that every perfectly normal space (regular is enough) with a countable base of open sets is separably metrizable (see [11]). We use both these equivalent formulations according to which is easier to deal with.

1. Some properties of spaces

Let $\mathcal{F}$ be a class of real-valued functions such that

(F1) all constant functions are in $\mathcal{F}$;

(F2) $(\forall f, g \in \mathcal{F})\ f - g \in \mathcal{F}$;

(F3) $(\forall f \in \mathcal{F})\ |f| \in \mathcal{F}$;

(F4) $(\forall f \in \mathcal{F})(\forall n > 0)\ f/n \in \mathcal{F}$.

For a topological space X we set $\mathcal{F}(X) = \mathcal{F} \cap {}^X\mathbb{R}$. Let $f, f_n : X \to \mathbb{R}$ for $n \in \omega$. We consider the following four kinds of convergence of the sequence $\{f_n\}_{n=0}^{\infty}$ to f:

(P) pointwise convergence on X,

(QN) quasi-normal convergence on X,

(Σ) $\sum_{n=0}^{\infty} |f_n(x) - f(x)| < \infty$ for $x \in X$,

(Σ^*) pseudo-normal convergence on X.

The sequence of functions in the above conditions may satisfy any of the following hypotheses ($\mathcal{F}$ is a given class of functions):

($\mathcal{F}$) $f_n \in \mathcal{F}(X)$ for $n \in \omega$, $f = 0$;

($\overline{\mathcal{F}}$) $f_n \in \mathcal{F}(X)$ for $n \in \omega$, f is arbitrary;

($\mathcal{F}^{\downarrow}$) $f_{n+1} \leqslant f_n$, $f_n \in \mathcal{F}(X)$ for $n \in \omega$, $f = 0$;

($\overline{\mathcal{F}}^{\downarrow}$) $f_{n+1} \leqslant f_n$, $f_n \in \mathcal{F}(X)$ for $n \in \omega$, f is arbitrary.

Definition 1.1. Let α be any of the hypotheses $\mathcal{F}$, $\overline{\mathcal{F}}$, $\mathcal{F}^{\downarrow}$, $\overline{\mathcal{F}}^{\downarrow}$ and let β, γ be any of the convergences P, QN, Σ, Σ^*.

(1) A space X is an $\alpha\beta\gamma$-space if whenever functions $f_n, f : X \to \mathbb{R}$, $n \in \omega$, satisfy condition (α), and the sequence $\{f_n\}_{n=0}^{\infty}$ β-converges to f, then the sequence $\{f_n\}_{n=0}^{\infty}$ γ-converges to f.

(2) A space X is a weak $\alpha\beta\gamma$-space (shortly w$\alpha\beta\gamma$-space) if whenever functions $f_n, f : X \to \mathbb{R}$, $n \in \omega$, satisfy condition (α), and the sequence $\{f_n\}_{n=0}^{\infty}$ β-converges to f, then there is an increasing sequence of integers $\{n_k\}_{k=0}^{\infty}$ such that the sequence $\{f_{n_k}\}_{k=0}^{\infty}$ γ-converges to f.

If $\mathcal{G}$ is another class of functions and $\mathcal{F}$ is closed under substitutions from $\mathcal{G}$ (i.e., whenever $g \in \mathcal{G}$, $g : X \to Y$, and $f \in \mathcal{F}(Y)$, then the composition $g \circ f$ belongs to $\mathcal{F}(X)$), then every of the defined properties is preserved by images of functions from $\mathcal{G}$. For example the Baire class of functions $\mathcal{B}_1(X)$ is closed under continuous substitutions and so a continuous image of a $\mathcal{B}_1$PQN-space is a $\mathcal{B}_1$PQN-space.

Lemma 1.2. *Every monotone Σ-convergent sequence of functions is* QN*-convergent.*

Proof. Let $f_n \in \mathcal{F}$, $f_{n+1} \leqslant f_n$, and $\sum_{n=0}^{\infty} |f_n(x) - f(x)| < \infty$ for $x \in X$. Then for all $x \in X$ for all but finitely many n we have $|f_n(x) - f(x)| < 1/\sqrt{n}$. Otherwise for infinitely many n we have $(\forall i \leqslant n)$ $|f_i(x) - f(x)| \geqslant |f_n(x) - f(x)| \geqslant 1/\sqrt{n}$ and so $\sum_{i=0}^{n} |f_i(x) - f(x)| \geqslant \sqrt{n}$. $\square$

Remark 1.3.

(a) Every $\alpha\beta\gamma$-space is a w$\alpha\beta\gamma$-space.

(b) Every space is an $\alpha\beta\beta$-space and a w$\alpha\beta\beta$-space (therefore we will consider only the possibilities with $\beta \neq \gamma$).

(c) Let $f_n = 1/(n+1)$ for $n \in \omega$ be constant functions. This sequence converges quasi-normally but does not Σ-converge. Therefore every non-empty space is neither an $\alpha\beta\Sigma$-space nor an $\alpha\beta\Sigma^*$-space for any α and for $\beta =$ P, QN.

(d) QN-convergence and Σ-convergence imply P-convergence. Hence every space is an $\alpha\beta$P-space and a w$\alpha\beta$P-space for $\beta =$ QN, Σ, Σ^*.

(e) Σ^*-convergence imply all other mentioned convergences. Therefore every space is an $\alpha\Sigma^*\beta$-space and a w$\alpha\Sigma^*\beta$-space for $\beta =$ QN, Σ.

(f) By Lemma 1.2 every space is an $\alpha\Sigma$QN-space, a w$\alpha\Sigma$QN-space, and a w$\alpha\Sigma\Sigma^*$-space for $\alpha = \mathcal{F}^{\downarrow}, \overline{\mathcal{F}}^{\downarrow}$.

(g) Every space is a wαQNΣ^*-space and a wαQNΣ-space for $\alpha = \mathcal{F}, \overline{\mathcal{F}}, \mathcal{F}^{\downarrow}, \overline{\mathcal{F}}^{\downarrow}$.

Now by (b) we consider $4 \cdot 4 \cdot 3 = 48$ possibilities in Definition 1.1(1) and the same number in Definition 1.1(2). By (c) we have to remove $4 \cdot 2 \cdot 2 = 16$ possibilities in Definition 1.1(1); by (d) we remove $4 \cdot 3 = 12$ possibilities in both definitions; by (e) we remove $4 \cdot 2 = 8$ possibilities in both definitions; by (f) we have to remove 2 possibilities in Definition 1.1(1) and 4 possibilities in Definition 1.1(2); by (g) we remove 8 possibilities in Definition 1.1(2). From this computation we can conclude that in Definitions 1.1(1) and 1.1(2) there are $10 = 48 - (16+12+8+2)$ and $16 = 48-(12+8+4+8)$ possibilities left, respectively. By the results presented below some of the so obtained properties coincide:

$$\mathcal{F}\mathrm{PQN},\ \overline{\mathcal{F}}\mathrm{PQN},\ \mathcal{F}\Sigma\mathrm{QN},\ \overline{\mathcal{F}}\Sigma\mathrm{QN},\ \mathcal{F}^{\downarrow}\Sigma\Sigma^*,\ \overline{\mathcal{F}}^{\downarrow}\Sigma\Sigma^*,$$

$$\mathcal{F}^{\downarrow}\mathrm{PQN} = \mathrm{w}\mathcal{F}^{\downarrow}\mathrm{PQN} = \mathrm{w}\mathcal{F}^{\downarrow}\mathrm{P}\Sigma^* = \mathrm{w}\mathcal{F}^{\downarrow}\mathrm{P}\Sigma \quad \text{(Lemma 1.5(1), (2))},$$

$$\overline{\mathcal{F}}^{\downarrow}\mathrm{PQN} = \mathrm{w}\overline{\mathcal{F}}^{\downarrow}\mathrm{PQN} = \mathrm{w}\overline{\mathcal{F}}^{\downarrow}\mathrm{P}\Sigma^* = \mathrm{w}\overline{\mathcal{F}}^{\downarrow}\mathrm{P}\Sigma \quad \text{(Lemma 1.5(1), (2))},$$

$$\mathcal{F}\Sigma\Sigma^* = \overline{\mathcal{F}}\Sigma\Sigma^* \quad \text{(Lemma 1.7)},$$

$$\mathrm{w}\mathcal{F}\mathrm{PQN} = \mathrm{w}\mathcal{F}\mathrm{P}\Sigma^* \quad \text{(Lemma 1.5(2))},$$

$$\mathrm{w}\overline{\mathcal{F}}\mathrm{PQN} = \mathrm{w}\overline{\mathcal{F}}\mathrm{P}\Sigma^* = \mathrm{w}\overline{\mathcal{F}}\mathrm{P}\Sigma \quad \text{(Lemma 1.9)},$$

$$\mathrm{w}\mathcal{F}\mathrm{P}\Sigma,$$

$$\mathrm{w}\mathcal{F}\Sigma\mathrm{QN} = \mathrm{w}\mathcal{F}\Sigma\Sigma^* \quad \text{(Lemma 1.5(3))},$$

$$\mathrm{w}\overline{\mathcal{F}}\Sigma\mathrm{QN} = \mathrm{w}\overline{\mathcal{F}}\Sigma\Sigma^* \quad \text{(Lemma 1.5(3))}.$$

Convention 1.4. Let us remark that the letter P is not used at the end of any prefix of $\alpha\beta\gamma$-space in the above possibilities. To simplify the notation we shall always write $\alpha\gamma$-space instead of αPγ-space and w$\alpha\gamma$-space instead of wαPγ-space.

Every space of cardinality less than $\mathfrak{b}$ is an $\overline{\mathcal{F}}$QN-space and (by Bartoszyński's characterization [1] of additivity of Lebesgue measure in terms of convergent series) every space of cardinality less than the additivity of Lebesgue measure is an $\mathcal{F}\Sigma\Sigma^*$-space.

Lemma 1.5.
(1) $\alpha\mathcal{QN} = \mathrm{w}\alpha\mathcal{QN} = \mathrm{w}\alpha\Sigma$ *for* $\alpha = \mathcal{F}^{\downarrow}, \overline{\mathcal{F}}^{\downarrow}$.
(2) $\mathrm{w}\alpha\mathcal{QN} = \mathrm{w}\alpha\Sigma^*$ *for* $\alpha = \mathcal{F}, \mathcal{F}^{\downarrow}, \overline{\mathcal{F}}, \overline{\mathcal{F}}^{\downarrow}$.
(3) $\mathrm{w}\alpha\Sigma\mathcal{QN} = \mathrm{w}\alpha\Sigma\Sigma^*$ *for* $\alpha = \mathcal{F}, \mathcal{F}^{\downarrow}, \overline{\mathcal{F}}, \overline{\mathcal{F}}^{\downarrow}$.

Proof. We prove part (1) of the lemma for $\alpha = \overline{\mathcal{F}}^{\downarrow}$. The other case is similar and parts (2) and (3) easily follow from definitions. Clearly, $\overline{\mathcal{F}}^{\downarrow}\mathcal{QN} \subseteq \mathrm{w}\overline{\mathcal{F}}^{\downarrow}\mathcal{QN} \subseteq \mathrm{w}\overline{\mathcal{F}}^{\downarrow}\Sigma$. For the inclusion $\mathrm{w}\overline{\mathcal{F}}^{\downarrow}\Sigma \subseteq \overline{\mathcal{F}}^{\downarrow}\mathcal{QN}$ assume that $f_n \in \mathcal{F}(X)$, $f_{n+1} \leqslant f_n$, and the functions f_n P-converge to f on X, where X is a $\mathrm{w}\overline{\mathcal{F}}^{\downarrow}\Sigma$-space. There is $\{n_k\}_{k=0}^{\infty}$ such that $\sum_{k=0}^{\infty}|f_{n_k}(x) - f(x)| < \infty$. Then $(\forall x)(\forall^{\infty}k)\ |f_{n_{k^2}}(x) - f(x)| < 1/(k+1)$, and so $f_{n_{k^2}}$ QN-converge to f. Otherwise there are infinitely many k such that $\sum_{i \leqslant k^2}|f_{n_i}(x) - f(x)| \geqslant k^2|f_{n_{k^2}}(x) - f(x)| \geqslant k^2/(k+1) \to \infty$ (therefore X is a $\mathrm{w}\overline{\mathcal{F}}^{\downarrow}$QN-space). Now let $\varepsilon_n = 1/(k+1)$, whenever $n_{k^2} \leqslant n < n_{(k+1)^2}$. Then for all $x \in X$, for all but finitely many n, $|f_n(x) - f(x)| \leqslant |f_{n_{k^2}}(x) - f(x)| < 1/(k+1) = \varepsilon_n$ and so X is an $\overline{\mathcal{F}}^{\downarrow}$QN-space. □

Lemma 1.6. $\mathcal{F}\Sigma\Sigma^* \subseteq \overline{\mathcal{F}}^{\downarrow}\mathcal{QN} \subseteq \overline{\mathcal{F}}\Sigma\mathcal{QN}$.

Proof. (1) Let $f_{n+1} \leqslant f_n$, $\lim_{n\to\infty} f_n(x) = f(x)$ for $x \in X$, and let X be an $\mathcal{F}\Sigma\Sigma^*$-space. Since $\sum_{n=0}^{\infty}(f_n(x) - f_{n+1}(x)) = f_0(x) - f(x) < \infty$, there is a convergent series of positive reals $\sum_{n=0}^{\infty}\varepsilon_n$ such that for every $x \in X$, $f_n(x) - f_{n+1}(x) \leqslant \varepsilon_n$ for all but finitely many n. Then $f_n(x) - f(x) \leqslant \sum_{k=n}^{\infty}\varepsilon_k$ for all but finitely many n.

(2) Let X be an $\overline{\mathcal{F}}^{\downarrow}$QN-space and let $\sum_{n=0}^{\infty}|f_n(x) - f(x)| < \infty$ for $x \in X$. Then $\sum_{n=0}^{\infty}|f_n(x) - f_{n+1}(x)| < \infty$. Set

$$g(x) = \sum_{m=0}^{\infty}\bigl|f_m(x) - f_{m+1}(x)\bigr|, \quad g_n(x) = \sum_{m=0}^{n-1}\bigl|f_m(x) - f_{m+1}(x)\bigr|.$$

There is $\{\varepsilon_n\}_{n=0}^{\infty}$ such that for all $x \in X$ for all but finitely many n we have $g(x) - g_n(x) < \varepsilon_n$, and, since $|f_n(x) - f(x)| = |\sum_{m=n}^{\infty}(f_m(x) - f_{m+1}(x))| \leqslant g(x) - g_n(x)$, we are done. □

Lemma 1.7. $\mathcal{F}\Sigma\Sigma^* = \overline{\mathcal{F}}\Sigma\Sigma^* \subseteq \overline{\mathcal{F}}^{\downarrow}\Sigma\Sigma^* \subseteq \mathcal{F}^{\downarrow}\Sigma\Sigma^*$.

Proof. It is enough to prove the inclusion $\mathcal{F}\Sigma\Sigma^* \subseteq \overline{\mathcal{F}}\Sigma\Sigma^*$. Let $f_n \in \mathcal{F}(X)$, $\sum_{n=0}^{\infty}|f_n(x) - f(x)| < \infty$ for all $x \in X$ and let X be an $\mathcal{F}\Sigma\Sigma^*$-space. By Lemma 1.6 there is a sequence of positive reals $\{\varepsilon'_n\}_{n=0}^{\infty}$ tending to 0 such that for every $x \in X$, $\sum_{m=n}^{\infty}|f_m(x) - f(x)| < \varepsilon'_n$ for all but finitely many n. Let $\varphi \in {}^{\omega}\omega$ be such that $\sum_{n=0}^{\infty}\varepsilon'_{\varphi(n)} < \infty$. Since

$$\sum_{n=0}^{\infty}\bigl|f_n(x) - f_{\varphi(n)}(x)\bigr| \leqslant \sum_{n=0}^{\infty}\bigl|f_{\varphi(n)}(x) - f(x)\bigr| + \sum_{n=0}^{\infty}\bigl|f_n(x) - f(x)\bigr| < \infty$$

and X is an $\mathcal{F}\Sigma\Sigma^*$-space there is a convergent series of positive reals $\sum_{n=0}^{\infty}\varepsilon''_n$ such that $|f_n(x)-f_{\varphi(n)}(x)|\leqslant\varepsilon''_n$ for all but finitely many n. Set $\varepsilon_n=\varepsilon''_n+\varepsilon'_{\varphi(n)}$. Then $\sum_{n=0}^{\infty}\varepsilon_n<\infty$ and $|f_n(x)-f(x)|\leqslant|f_n(x)-f_{\varphi(n)(x)}|+|f_{\varphi(n)}(x)-f(x)|\leqslant\varepsilon_n$ for all but finitely many n. □

Lemma 1.8. $\mathrm{w}\mathcal{F}\Sigma\subseteq\mathcal{F}^{\downarrow}\mathcal{QN}$, $\mathrm{w}\overline{\mathcal{F}}\Sigma\subseteq\overline{\mathcal{F}}^{\downarrow}\mathcal{QN}$.

Proof. Let $f_n\in\mathcal{F}(X)$, $f_{n+1}\leqslant f_n$, and $f(x)=\lim_{n\to\infty}f_n(x)$ on X, where X is a $\mathrm{w}\overline{\mathcal{F}}\Sigma$-space (respectively $\mathrm{w}\mathcal{F}\Sigma$-space and $f=0$). Let $\{n_k\}_{k=0}^{\infty}$ be an increasing sequence of integers such that $\sum_{k=0}^{\infty}(f_{n_k}(x)-f(x))<\infty$ on X. By Remark 1.3(f) every space is an $\overline{\mathcal{F}}^{\downarrow}\Sigma$QN-space and so f_{n_k} QN-converge to f. Therefore X is a $\mathrm{w}\overline{\mathcal{F}}^{\downarrow}$QN-space (respectively $\mathrm{w}\mathcal{F}^{\downarrow}$QN-space) which by Lemma 1.5(1) has the same meaning as an $\overline{\mathcal{F}}^{\downarrow}$QN-space (respectively $\mathcal{F}^{\downarrow}$QN-space). □

Lemma 1.9. $\mathrm{w}\overline{\mathcal{F}}\mathcal{QN}=\mathrm{w}\overline{\mathcal{F}}\Sigma^*=\mathrm{w}\overline{\mathcal{F}}\Sigma$.

Proof. The inclusions $\mathrm{w}\overline{\mathcal{F}}\mathcal{QN}\subseteq\mathrm{w}\overline{\mathcal{F}}\Sigma^*\subseteq\mathrm{w}\overline{\mathcal{F}}\Sigma$ are trivial (Lemma 1.5(2)). Clearly, $\mathrm{w}\overline{\mathcal{F}}\mathcal{QN}=\mathrm{w}\overline{\mathcal{F}}\Sigma\cap\mathrm{w}\overline{\mathcal{F}}\Sigma\mathcal{QN}$ and by Lemmas 1.8 and 1.6 we have $\mathrm{w}\overline{\mathcal{F}}\Sigma\subseteq\overline{\mathcal{F}}^{\downarrow}\mathcal{QN}\subseteq\overline{\mathcal{F}}\Sigma\mathcal{QN}\subseteq\mathrm{w}\overline{\mathcal{F}}\Sigma\mathcal{QN}$. Therefore $\mathrm{w}\overline{\mathcal{F}}\mathcal{QN}=\mathrm{w}\overline{\mathcal{F}}\Sigma$. □

Summarizing the results of this section we obtain Diagram 1 in which all so far known inclusions between the introduced classes are presented.

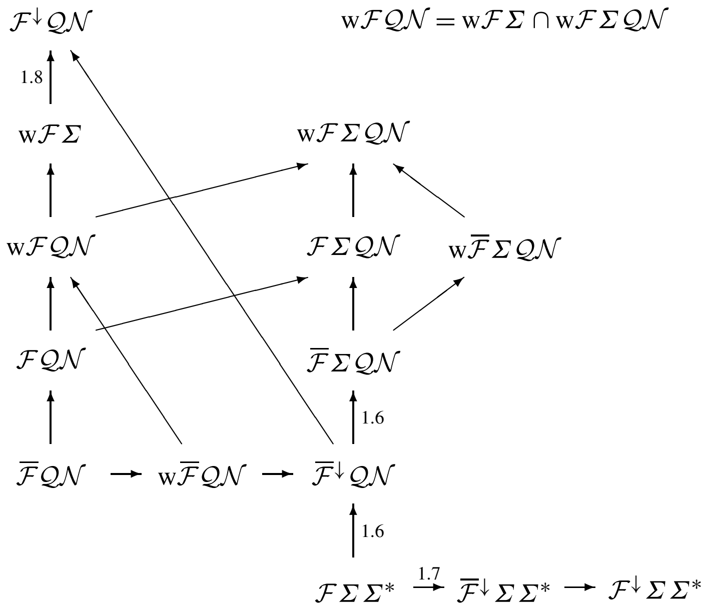

Diagram 1. This diagram shows all inclusions between the classes of spaces with introduced properties which we prove in Section 1. The numbers at the arrows refer to the lemmas in which proofs are given. The other inclusions are easy consequences of definitions. For additional inclusion $\overline{\mathcal{F}}^{\downarrow}\Sigma\Sigma^*\subseteq\overline{\mathcal{F}}^{\downarrow}\mathcal{QN}$ see [25].

2. Borel measurable functions

It is very natural to examine the classes of sets introduced in the previous section by taking $\mathcal{F} = \mathcal{C}$ or $\mathcal{F} = \mathcal{M}$, the class of all continuous real-valued functions and the class of all Borel measurable real-valued functions, respectively. In the case of $\mathcal{F} = \mathcal{M}$, if $f_n \in \mathcal{M}(X)$ and f_n converge to f, then $f_n - f \in \mathcal{M}(X)$ and $f_n - f$ converge to 0. Moreover, if we define

$$f'_n(x) = \sup\{|f_k(x) - f(x)|: k \geqslant n\},$$

then $f'_n \in \mathcal{M}(X)$, $f'_{n+1} \leqslant f'_n$, f'_n converge to 0 and, for example, f'_n QN-converge to 0 if and only if f_n QN-converge to f. Therefore one can easily see that

$$\overline{\mathcal{M}}\mathcal{QN} = \mathcal{M}^{\downarrow}\mathcal{QN}, \qquad \overline{\mathcal{M}}\Sigma\mathcal{QN} = \mathcal{M}\Sigma\mathcal{QN},$$
$$\mathrm{w}\overline{\mathcal{M}}\Sigma\mathcal{QN} = \mathrm{w}\mathcal{M}\Sigma\mathcal{QN}, \quad \text{and} \quad \overline{\mathcal{M}}^{\downarrow}\Sigma\Sigma^* = \mathcal{M}^{\downarrow}\Sigma\Sigma^*.$$

In [25] it is proved that $\mathcal{M}\Sigma\Sigma^* = \mathcal{M}^{\downarrow}\Sigma\Sigma^*$ (see also Theorem 6.2(ii)). Consequently, in the case $\mathcal{F} = \mathcal{M}$, each class of Diagram 1 coincides with one of the four classes of this simple diagram:

$$\mathcal{M}\Sigma\Sigma^* \to \mathcal{M}\mathcal{QN} \to \mathcal{M}\Sigma\mathcal{QN} \to \mathrm{w}\mathcal{M}\Sigma\mathcal{QN}.$$

Let $\mathcal{A}$ be a family of sets. We say that $\mathcal{A}$ *is weakly distributive on a set* X if whenever $A_{n,m} \in \mathcal{A}$ for $n, m \in \omega$ are such that $X \subseteq \bigcap_{n=0}^{\infty}\bigcup_{m=0}^{\infty} A_{n,m}$, then there is a function $\varphi \in {}^{\omega}\omega$ such that $X \subseteq \bigcup_{k=0}^{\infty}\bigcap_{n\geqslant k}\bigcup_{m\leqslant\varphi(n)} A_{n,m}$. We say that $\mathcal{A}$ satisfies *the σ-reduction theorem* if for every sequence of sets A_n, $n \in \omega$ there are pairwise disjoint sets $A'_n \in \mathcal{A}$ such that $A'_n \subseteq A_n$ and $\bigcup_{n=0}^{\infty} A'_n = \bigcup_{n=0}^{\infty} A_n$. The family of Borel sets and the family of F_σ sets of a perfectly normal space satisfy the σ-reduction theorem [20]. The following theorem is implicitly in [6].

Theorem 2.1. *Let $\mathcal{A}$ be a family of subsets of X.*

(1) *$\mathcal{A}$ is weakly distributive on X if and only if $\mathcal{A}_\sigma$ is weakly distributive on X ($\mathcal{A}_\sigma$ is a family of countable unions of sets from $\mathcal{A}$).*
(2) *If $\mathcal{A}$ is weakly distributive on X, then for every $\mathcal{A}_\sigma$-measurable function $f : X \to {}^{\omega}\omega$ the image $f(X)$ is bounded.*
(3) *Let $\mathcal{A} \subseteq \mathcal{P}(X)$ satisfy the σ-reduction theorem. Then all $\mathcal{A}$-measurable images of X into ${}^{\omega}\omega$ are bounded if and only if $\mathcal{A}$ is weakly distributive on X.*
(4) *If $\mathcal{A}$ is a σ-algebra which is weakly distributive on X, then*
    (a) *$\mathcal{A}$ is weakly distributive on every $Y \subseteq X$,*
    (b) *for every $\mathcal{A}{\restriction}Y$-measurable function $f : Y \to {}^{\omega}\omega$ the image $f(Y)$ is bounded (let us recall that $\mathcal{A}{\restriction}Y = \{A \cap Y: A \in \mathcal{A}\}$),*
    (c) *every set $Y \subseteq X$ is an $\mathcal{F}_{\mathcal{A}}$QN-space, i.e., X is a hereditary $\mathcal{F}_{\mathcal{A}}$QN-space ($\mathcal{F}_{\mathcal{A}}$ denotes the family of all $\mathcal{A}{\restriction}Y$-measurable functions $f : Y \to \mathbb{R}$ for $Y \subseteq X$).*

Proof. Condition (1) is trivial.

(2) We can assume that $\mathcal{A}=\mathcal{A}_\sigma$. If $f:X\to {}^\omega\omega$ is $\mathcal{A}$-measurable, we set $A_{n,m}=\{x\in X:\ f(x)(n)=m\}$ and use weak distributivity of $\mathcal{A}$.

(3) Let $X\subseteq\bigcap_{n=0}^\infty\bigcup_{m=0}^\infty A_{n,m}$. By σ-reduction theorem we can find $A'_{n,m}\subseteq A_{n,m}$, $A'_{n,m}\in\mathcal{A}$, such that $A'_{n,m_1}\cap A'_{n,m_2}=\emptyset$ for $m_1\neq m_2$, and $\bigcup_{m=0}^\infty A'_{n,m}=\bigcup_{m=0}^\infty A_{n,m}$ for every n. Let us define an $\mathcal{A}$-measurable function $f:X\to{}^\omega\omega$ by $f(x)=\alpha$ if and only if $x\in\bigcap_{n=0}^\infty A'_{n,\alpha(n)}$. There is a function $\varphi\in{}^\omega\omega$ such that $(\forall x\in X)(\forall^\infty n)\ f(x)(n)\leqslant\varphi(n)$. Then

$$X\subseteq\bigcup_{k=0}^\infty\bigcap_{n\geqslant k}\bigcup_{m\leqslant\varphi(n)}A'_{n,m}\subseteq\bigcup_{k=0}^\infty\bigcap_{n\geqslant k}\bigcup_{m\leqslant\varphi(n)}A_{n,m}.$$

(4) Let $f:Y\to{}^\omega\omega$ be $\mathcal{A}\restriction Y$-measurable. There is an $\mathcal{A}$-measurable function $f':X\to{}^\omega\omega$ such that $f\subseteq f'$. By (2) it follows that $f'(X)$ is bounded and consequently also $f(Y)$ is bounded and by (3) $\mathcal{A}\restriction Y$ is weakly distributive.

Finally let $f_n:Y\to\mathbb{R}$, $n\in\omega$ be a sequence of $\mathcal{A}\restriction Y$-measurable functions such that f_n converge to 0. The sets $A_{n,m}=\{x\in Y:\ (\forall k\geqslant m)\ f_k(x)<1/(n+1)\}$ are in $\mathcal{A}\restriction Y$ hence by weak distributivity of $\mathcal{A}\restriction Y$ we get $\varphi\in{}^\omega\omega$ such that $(\forall x\in Y)(\forall^\infty n)(\forall k\geqslant\varphi(n))$ $f_k(x)<1/(n+1)$. Without loss of generality we can assume that $\varphi(n)<\varphi(n+1)$ for all n. Let us define $\varepsilon_k=1/(n+1)$, whenever $\varphi(n)\leqslant k<\varphi(n+1)$. Now $\lim_{k\to\infty}\varepsilon_k=0$ and the sequence $\{\varepsilon_n\}_{n=0}^\infty$ witnesses that f_n QN-converge to 0. Therefore Y is an $\mathcal{F}_{\mathcal{A}}$QN-space. □

In the next sections we are interested in the case $\mathcal{C}\subseteq\mathcal{F}\subseteq\mathcal{M}$ and mainly in $\mathcal{F}=\mathcal{C}$. In the case $\mathcal{F}=\mathcal{C}$ we introduce a shorter notation which in particular cases coincides with the notation introduced in [6].

Definition 2.2.

(1) An mQN-space ($\overline{\text{m}}$QN-space) is a $\mathcal{C}^\downarrow$QN-space ($\overline{\mathcal{C}}^\downarrow$QN-space).
(2) A QN-space (wQN-space) is a $\mathcal{C}$QN-space (w$\mathcal{C}$QN-space).
(3) A ΣQN-space (wΣQN-space) is a $\mathcal{C}\Sigma$QN-space (w$\mathcal{C}\Sigma$QN-space).
(4) A $\overline{\text{QN}}$-space is a $\overline{\mathcal{C}}$QN-space.
(5) A $\overline{\Sigma}$QN-space (w$\overline{\Sigma}$QN-space) is a $\overline{\mathcal{C}}\Sigma$QN-space (w$\overline{\mathcal{C}}\Sigma$QN-space).
(6) A Σ-space is a w$\mathcal{C}\Sigma$-space.
(7) A $\Sigma\Sigma^*$-space (m$\Sigma\Sigma^*$-space, $\overline{\text{m}}\Sigma\Sigma^*$-space) is a $\mathcal{C}\Sigma\Sigma^*$-space ($\mathcal{C}^\downarrow\Sigma\Sigma^*$-space, $\overline{\mathcal{C}}^\downarrow\Sigma\Sigma^*$-space).

Theorem 2.3. *Each of the properties in Definition 2.2 is σ-additive, and for perfectly normal spaces, is hereditary for F_σ subsets, and is preserved by $\mathcal{D}_1$ images.*

Proof. We prove the closure for closed subsets and by σ-additivity we get the closure for F_σ subsets. In perfectly normal spaces every closed set is a G_δ set and every continuous function defined on a closed subset F of X can be continuously extended to a continuous function defined on X. So if $f_n:F\to\mathbb{R}$, $n\in\omega$ is a sequence of continuous functions with F a closed subset of X we can find open sets U_n for $n\in\omega$ such that $F=\bigcap_{n=0}^\infty U_n$ and there are continuous functions $f'_n : X \to \mathbb{R}$ such that $f'_n \restriction F = f \restriction F$ and $f'_n \restriction (X \setminus U_n) = 0$. Then f'_n converge discretely to 0 on the set $X \setminus F$ (moreover, if $f_{n+1} \leqslant f_n$ for all n, we can take instead f'_n the functions $f''_n = \min\{f'_k\colon k \leqslant n\}$). Consequently, if f_n, $n \in \omega$ satisfies some of the considered hypotheses P, QN, Σ on the set F, then so does the sequence of f'_n (and the sequence of f''_n). Now using the appropriate property of the space X we can derive for these sequences of functions required convergences on the set X and consequently also on the set F.

Let $f \in \mathcal{D}_1(X, Y)$, i.e., there is a sequence of functions $f_n \in \mathcal{C}(X, Y)$ such that $(\forall x \in X)$ $(\forall^\infty n)$ $f_n(x) = f(x)$. The sets

$$F_m = \bigl\{x \in X\colon (\forall n \geqslant m)\ f_n(x) = f(x)\bigr\}$$

are closed and since X is perfectly normal F_m possess the same convergence property as X does. Therefore $f(F_m) = f_m(F_m)$ has the same property and so also $f(X)$ by σ-additivity. □

All the proofs of the relationship between the classes of spaces presented so far essentially use the algebraic properties (F1)–(F4) only. From now on the most of the results require a topological structure on the set X, deriving the family $\mathcal{F}$ from the topology. The usual assumption in such results is that X is a perfectly normal space. The next result serves as an example.

Theorem 2.4. *Let X be a perfectly normal space.*

(1) *X is an* $\mathrm{m}\Sigma\Sigma^*$*-space if and only if X is an* $\overline{\mathrm{m}}\Sigma\Sigma^*$*-space.*

(2) *X is a* $\overline{\Sigma}$QN*-space if and only if X is a* w$\overline{\Sigma}$QN*-space and a* ΣQN*-space.*

Proof. (1) Let X be an $\mathrm{m}\Sigma\Sigma^*$-space and let $f_{n+1} \leqslant f_n$ for $n \in \omega$ be continuous functions, $f(x) = \lim_{n\to\infty} f_n(x)$ for $x \in X$ and let $\sum_{n=0}^{\infty}(f_n(x) - f(x)) < \infty$ for $x \in X$. By Lemma 1.2 this sequence quasi-normally converges and so $X = \bigcup_{k=0}^{\infty} F_k$ with F_k closed and such that the convergence is uniform on each set F_k. Therefore $(f_n - f)\restriction F_k$ are continuous and as by Theorem 2.3 all closed subsets of X are $\mathrm{m}\Sigma\Sigma^*$-spaces the sequence of functions $\{f_n - f\}_{n=0}^{\infty}$ does Σ^*-converge on each set F_k. Each Σ^*-convergence is witnessed by a convergent series of reals. Now as countable family of convergent series can be majorized by a single convergent series we easily obtain Σ^*-convergence of the sequence of functions on the whole space X.

(2) Let X be a w$\overline{\Sigma}$QN-space and a ΣQN-space and let $f_n \in \mathcal{C}(X)$ for $n \in \omega$, and $f(x) = \lim_{n\to\infty} f_n(x)$ be such that $\sum_{n=0}^{\infty}|f_n(x) - f(x)| < \infty$. There is an increasing sequence of integers $\{n_k\}_{k=0}^{\infty}$ such that f_{n_k} QN-converge to f. Therefore $X = \bigcup_{k=0}^{\infty} F_k$ with F_k closed and $f \restriction F_k$ continuous for every k. Since F_k is also a ΣQN-space, the sequence $\{f_n \restriction F_k\}_{n=0}^{\infty}$ QN-converges to $f \restriction F_k$ for all $n \in \omega$. Now it is easy to prove that also f_n QN-converge to f. □

3. mQN-space

We say that a space X is *nestled* if for every sequence of strictly positive functions $f_n \in \mathcal{C}(X)$ there is a sequence of positive integers k_n such that

$$(\forall x \in X)(\forall^\infty n)(\forall i \leqslant n) \quad f_i(x) > 1/k_n.$$

Lemma 3.1. *Let X be arbitrary space. The following conditions are equivalent.*

(1) *X is a nestled space.*
(2) *Every continuous image of X into ${}^\omega\mathbb{R}$ is eventually bounded.*
(3) *For every sequence of strictly positive functions $f_n \in \mathcal{C}(X)$, $n \in \omega$ there are integers m_n such that $(\forall x \in X)(\forall^\infty n)$ $f_{n+1}(x)/m_{n+1} < f_n(x)/m_n$.*

Proof. (1) $\Rightarrow$ (2) Let $\varphi : X \to {}^\omega\mathbb{R}$ be a continuous function. The functions $f_n(x) = 1/(|\varphi(x)(n)| + 1)$ are continuous. There are integers k_n such that for every $x \in X$ for all but finitely many n we have $f_n(x) > 1/k_n$ and hence $|\varphi(x)(n)| < k_n$.

(2) $\Rightarrow$ (3) If f_n are strictly positive continuous functions the mapping $\varphi : X \to {}^\omega\mathbb{R}$ defined by $\varphi(x)(n) = f_{n+1}(x)/f_n(x)$ is continuous and $\varphi(X)$ is eventually bounded by a function ψ. Let us set $m_n = \prod_{i<n} \psi(i)$.

(3) $\Rightarrow$ (1) We define $f'_n(x) = 1/\prod_{i<n} f_i(x)$. Let m_n be integers such that $(\forall x \in X)$ $(\forall^\infty n)$ $f'_{n+1}(x)/m_{n+1} < f'_n(x)/m_n$. There are k'_n such that $1/k'_n < m_n/m_{n+1}$. Then $(\forall x \in X)(\forall^\infty n)$ $f_n(x) = f'_n(x)/f'_{n+1}(x) > m_n/m_{n+1} > 1/k'_n$. To obtain the stronger property in the definition of nestled space it is enough to take $k_n = \max\{k'_i \colon i \leqslant n\} + n$. □

Lemma 3.2. *Let X be an* mQN*-space, or a nestled space. Then* Clopen(X) *is weakly distributive on X and consequently every continuous image of X into ${}^\omega\omega$ is bounded.*

Proof. Let $X = \bigcap_{n=0}^\infty \bigcup_{m=0}^\infty A_{n,m}$, with $A_{n,m}$ clopen. We can assume that $A_{n,m_1} \cap A_{n,m_2} = \emptyset$ for $m_1 \neq m_2$ (otherwise take $A'_{n,m} = A_{n,m} \setminus \bigcup_{i<m} A_{n,i}$). Let us define

$$f_k(x) = 1/\left(\min\left\{n\colon x \notin \bigcup_{m<k-n} A_{n,m}\right\} + 1\right),$$
$$g_n(x) = 1/(m+1), \quad \text{for } x \in A_{n,m}.$$

The functions f_k, g_n are continuous, $f_{k+1}(x) \leqslant f_k(x)$, $\lim_{k\to\infty} f_k(x) = 0$, and $g_n(x) > 0$ for $x \in X$. If X is an mQN-space, then there exists a decreasing sequence of positive reals $\{\varepsilon_n\}_{n=0}^\infty$ converging to 0 such that $(\forall x \in X)(\forall^\infty k)$ $f_k(x) \leqslant \varepsilon_k$. Let $\varphi \in {}^\omega\omega$ be increasing such that $\varepsilon_{\varphi(n)} < 1/(n+1)$. Then for every $x \in X$ for all but finitely many n, $f_{\varphi(n)}(x) \leqslant \varepsilon_{\varphi(n)} < 1/(n+1)$ and so $x \in \bigcup_{m<\varphi(n)-n} A_{n,m}$. If X is a nestled space, there is $\psi \in {}^\omega\omega$ such that $(\forall x \in X)(\forall^\infty n)$ $f_n(x) > 1/\psi(n)$. Then for every $x \in X$ for all but finitely many n, $x \in \bigcup_{m<\psi(n)} A_{n,m}$.

The second part is a consequence of Theorem 2.1. □

Lemma 3.3. *Assume that* Open(X) *is weakly distributive. Then*

(1) *X is an* mQN*-space, and*

(2) *X is a nestled space.*

In particular, every countably compact topological space is an mQN-*space and a nestled space.*

Proof. (1) Let $f_n \in \mathcal{C}(X)$, $n \in \omega$ be a decreasing sequence of functions converging to 0 on X. Let us set $A_{n,m} = \{x \in X\colon\ f_m(x) < 1/(n+1)\}$. By open weak distributivity there is a function $\varphi \in {}^{\omega}\omega$ such that $X \subseteq \bigcup_{k=0}^{\infty} \bigcap_{n \geqslant k} A_{n,\varphi(n)}$. We can assume that φ is strictly increasing. Let us define $\varepsilon_k = 1/(n+1)$, whenever $\varphi(n) \leqslant k < \varphi(n+1)$. Clearly, the sequence $\{\varepsilon_n\}_{n=0}^{\infty}$ witnesses the quasi-normal convergence of functions f_n on X.

(2) For $A_{n,m} = \{x \in X\colon\ (\forall i \leqslant n)\ f_i(x) > 1/(m+1)\}$, $X = \bigcap_{n=0}^{\infty} \bigcup_{m=0}^{\infty} A_{n,m}$ and by open distributivity there is $\varphi \in {}^{\omega}\omega$ such that $X = \bigcup_{k=0}^{\infty} \bigcap_{n=k}^{\infty} A_{n,\varphi(n)}$. Let us set $k_n = \varphi(n) + 1$. □

Lemma 3.4. *Let X be a perfectly normal space and let $A \subseteq X$ be nestled. If $F \subseteq X \setminus A$ is an F_σ set, then there is a G_δ set G such that $F \subseteq G \subseteq \mathrm{cl}(F) \setminus A$. Consequently, $A \cap \mathrm{cl}(F)$ is meager in* $\mathrm{cl}(F)$.

Proof. Let $F = \bigcup_{n=0}^{\infty} F_n$ with F_n closed. For $n \in \omega$ let $g_n : X \to [0,1]$ be a continuous function such that $F_n = g_n^{-1}(\{0\})$. Let $A_{n,m} = \{x \in X\colon\ (\forall i \leqslant n)\ g_i(x) \geqslant 1/(m+1)\}$. Since A is nestled there are integers k_n such that $A \subseteq \bigcup_{m=0}^{\infty} \bigcap_{n=m}^{\infty} A_{n,k_n}$. Let us set $G = \bigcap_{m=0}^{\infty} (\mathrm{cl}(F) \setminus N_m)$, where $N_m = \bigcap_{n=m}^{\infty} A_{n,k_n}$ are closed sets disjoint with F. □

Every σ-compact perfectly normal space fulfills the assumptions of the next assertion.

Theorem 3.5. *Let X be a perfectly normal space such that* $\mathrm{Open}(X)$ *is weakly distributive on X. For every set $A \subseteq X$ the following conditions are equivalent.*

(1) $\mathrm{Open}(X)$ *is weakly distributive on A.*

(2) *A is a nestled space.*

(3) *For every G_δ set G containing A there exists an F_σ set F such that $A \subseteq F \subseteq G$.*

Proof. The implications (1) ⇒ (2) ⇒ (3) hold true by Lemmas 3.3 and 3.4.

(3) ⇒ (1) Let $A \subseteq G$, where $G = \bigcap_{n=0}^{\infty} \bigcup_{m=0}^{\infty} U_{n,m}$ with $U_{n,m}$ open. Let F be an F_σ set such that $A \subseteq F \subseteq G$. As an open distributivity is hereditary for F_σ subsets there is $\varphi \in {}^{\omega}\omega$ such that $F \subseteq \bigcup_{k=0}^{\infty} \bigcap_{n=k}^{\infty} \bigcup_{m \leqslant \varphi(n)} U_{n,m}$. Therefore $\mathrm{Open}(X)$ is weakly distributive on A. □

Theorem 3.6. *A separable metric* mQN-*space X is the union of countably many totally bounded subspaces. Consequently, every* mQN-*subset of a Polish space can be covered by a σ-compact set.*

Proof. Let $\{r_n\colon\ n \in \omega\}$ be a countable dense subset of X. Let $Q_n = \{r_k\colon\ k \leqslant n\}$. The functions $f_n(x) = d(x, Q_n)$ are continuous converging to 0 on X and $f_{n+1} \leqslant f_n$. Hence there is a sequence of positive reals ε_n converging to 0 such that for every $x \in X$,

$(\forall^\infty n)\, d(x, Q_n) \leqslant \varepsilon_n$. The sets $X_k = \{x \in X\colon\ (\forall n \geqslant k)\, d(x, Q_n) \leqslant \varepsilon_n\}$ are totally bounded and $X = \bigcup_{k=0}^{\infty} X_k$. □

By the same proof as that of Theorem 3.5 (replacing Lemma 3.4 by Theorem 3.6 in the argument—let us recall that G_δ subsets of a Polish space are again Polish spaces and open sets are weakly distributive on σ-compact sets) we obtain the following characterization.

Theorem 3.7. *Let X be a Polish space. For every set $A \subseteq X$ the following conditions are equivalent.*

(1) $\mathrm{Open}(X)$ *is weakly distributive on A.*
(2) *A is an* mQN-*space.*
(3) *For every G_δ set G containing A there exists a σ-compact set F such that $A \subseteq F \subseteq G$.*

Clopen sets are trivially weakly distributive in a connected space while open sets need not. There is a separable connected space which is not an mQN-space. Actually, let $Q \subseteq [0, 1] \times [0, 1]$ be a countable dense subset, then by Theorem 3.7 the connected set $X = ([0, 1] \times [0, 1]) \setminus Q$ is not an mQN-space.

We say that a space X is *perfectly meager* if every perfect set P in X is meager in the relativized topology of P. Notice that if $X \subseteq Y$, Y has a countable base of open sets, and X is perfectly meager, then for every perfect set $P \subseteq Y$ the set $X \cap P$ is meager in P.

Theorem 3.8.

(1) *A separable metric space is an* mQN-*space if and only if open sets are weakly distributive.*
(2) *A separable metric* mQN-*space is a nestled space.*
(3) *In a σ-compact perfectly normal space every nestled subset is an* mQN-*space.*
(4) *In a σ-compact metric space the* mQN-*subsets coincide with the nestled subsets.*
(5) *If X is a separable metric* mQN-*space all compact subsets of which are countable, then X is perfectly meager.*
(6) *Let Y be a perfectly normal hereditary separable σ-compact space and let X be a subspace of Y. If* $\mathrm{Open}(X)$ *is weakly distributive and every compact subset of X is countable, then X is perfectly meager.*

Proof. A separable metric space can be embedded into a Polish space. Therefore the assertions (1)–(4) easily follow from Theorems 3.5 and 3.7.

(5), (6) Let P be a perfect set in X. P can be embedded onto a dense subset of a space $\overline{P}$, which is either a Polish space, in case (5), or a perfectly normal hereditary separable σ-compact space, in case (6). Let Q be a countable dense subset of $\overline{P}$ disjoint with P. In the metric case by Theorem 3.7 open sets are weakly distributive on X and hence in both cases open sets are weakly distributive on a closed subset P of X. Therefore by Lemma 3.4 P is meager in $\overline{P}$ (and also relatively in P). □

The equivalence (4) ≡ (5) in the following theorem is due to Hurewicz.

Theorem 3.9. *Assume that every open set in a space X is a countable union of clopen sets. Then the following conditions are equivalent.*

(1) Clopen(X) *is weakly distributive.*
(2) *X is an* mQN-*space.*
(3) *X is a nestled space.*
(4) Open(X) *is weakly distributive.*
(5) *Every continuous image of X into ${}^{\omega}\omega$ is bounded.*
(6) *Every $\mathcal{D}_1$-image of X into a Polish space Y is a subset of a σ-compact subset of Y.*

Proof. The equivalence (1) $\equiv$ (4) is an easy consequence of the hypotheses, and the equivalence (1) $\equiv$ (5) holds true by Theorem 2.1(2) and (3). Therefore by Lemmas 3.2 and 3.3, we have (1) $\equiv$ (2) $\equiv$ (3) $\equiv$ (4) $\equiv$ (5). The implication (6) $\Rightarrow$ (5) is trivial. We prove (2) $\Rightarrow$ (6). Let $f \in \mathcal{D}_1(X, Y)$. There are closed sets $X_n \subseteq X$ for $n \in \omega$ such that $X = \bigcup_{n=0}^{\infty} X_n$ and $f \upharpoonright X_n$ is continuous for each n. Since X is an mQN-space, all sets X_n and consequently also $f(X_n)$ are mQN-spaces. Therefore $f(X) = \bigcup_{n=0}^{\infty} f(X_n)$ is an mQN-subset of a Polish space. By Theorem 3.6, $f(X)$ is a subset of a σ-compact set. □

Problem 3.10 (Lemma 3.3, Theorems 3.5, 3.7 and 3.8(1)). Is there a perfectly normal (nestled) mQN-space X such that Open(X) is not weakly distributive?

Problem 3.11.

(1) Does the conclusion of Lemma 3.4 hold true for mQN-subspaces too?
(2) (Theorem 3.8) Are the notions of an mQN-space and a nestled space different?
(3) (Theorem 3.5) Can every perfectly normal space be embedded into a perfectly normal space open sets of which are weakly distributive?

A space X is said to be a *σ-space* if every G_δ set in X is an F_σ set.

Theorem 3.12. *Let X be a perfectly normal space. The following conditions are equivalent.*

(1) *X is a hereditary* mQN-*space.*
(2) *X is a σ-space and* Open(X) *is weakly distributive.*
(3) *X is a σ-space and an* mQN-*space.*
(4) Open(X) *is weakly distributive on all subsets of X.*

Proof. (1) $\Rightarrow$ (2) If X is a hereditary mQN-space, then for every continuous function $f : X \to \mathbb{R}$ the image $f(X)$ is also hereditary mQN-space, and in particular, $f(X)$ is totally disconnected. Therefore, as X is perfectly normal, and every open set is a co-zero set, every open set is a countable union of clopen sets, and by Theorem 3.9, Open(X) is weakly distributive. By Lemma 3.4 every G_δ mQN-set in a perfectly normal space is an F_σ set. Therefore a hereditary mQN-space is a σ-space.

(3) $\Rightarrow$ (1) Let $Y \subseteq X$ and let $f_n \in \mathcal{C}(Y)$, $n \in \omega$ be a decreasing sequence of continuous functions converging to 0 on Y. The functions f_n can be extended to some functions f_n' all

continuous on a G_δ set $G \subseteq X$ with $Y \subseteq G$. The set $G' = \{x \in G\colon (\forall n)\ f_{n+1}(x) \leqslant f_n(x)\}$ is relatively closed in G and since X is perfectly normal it is still a G_δ set in X. Now as X is a σ space G' is an F_σ set and consequently an mQN-space. Therefore f_n' QN-converge to 0 on G' and Y is an mQN-space.

The implication (4) $\Rightarrow$ (1) is trivial, the implication (2) $\Rightarrow$ (3) holds true by Lemma 3.3, and (2) $\Rightarrow$ (4) is due to the fact that open distributivity is hereditary for F_σ subsets. □

4. mQN **is not Luzin**

Let X be a topological space, κ an infinite cardinal, $Y, A \subseteq X$. We say that Y is *κ-concentrated on a set* A if for every open set $U \supseteq A$, $|Y \setminus U| < \kappa$. A space X is a *κ-Luzin space* if $|X| \geqslant \kappa$ and every meager subset of X has size $< \kappa$ (for $\kappa = \omega_1$ it is called also a ν-space, see [20]). A set $A \subseteq X$ is a *κ-Luzin set in* X if $|A| \geqslant \kappa$ and $|A \cap B| < \kappa$ for every meager set $B \subseteq X$. Every κ-Luzin set is a κ-Luzin space and if $A \subseteq X$ with the relativized topology is a κ-Luzin space, then A is a κ-Luzin set in $\mathrm{cl}(A)$. Every separable metric κ-Luzin space is homeomorphic to a κ-Luzin subset of the Baire space ${}^{\omega}\omega$ [20].

Theorem 4.1. *Let X be a perfectly normal space. If* $\mathrm{cf}(\kappa) \geqslant \omega_1$ *and X is κ-concentrated on a countable subset A, then $|Y| < \kappa$ for every* mQN-*set $Y \subseteq X \setminus A$. In particular, if $|X| \geqslant \mathfrak{b}$, then $\kappa \geqslant \mathfrak{b}$.*

Proof. Every image of X by a continuous function $g : X \to \mathbb{R}$ is κ-concentrated on the countable set $g(A)$, and hence it is a zero-dimensional subset of $\mathbb{R}$. Every open set in X is a co-zero set of a continuous function and hence it is a countable union of clopen sets.

Let $\langle a_n\colon n \in \omega\rangle$ be an enumeration of the set A with infinitely many repetitions. By the previous part for each $n \in \omega$ we can find clopen sets $V_{n,m}$ for $m \in \omega$ such that $X \setminus \{a_n\} = \bigcup_{m=0}^{\infty} V_{n,m}$ and by clopen distributivity on Y there is $\varphi \in {}^{\omega}\omega$ such that $Y \subseteq \bigcup_{k=0}^{\infty} B_k$, where $B_k = \bigcap_{n=k}^{\infty} \bigcup_{m \leqslant \varphi(n)} V_{n,m}$ are closed. As $A \cap B_k = \emptyset$ it follows that $|B_k| < \kappa$, and hence $|Y| < \kappa$. Finally let us recall that every set Y of cardinality $< \mathfrak{b}$ is an mQN-set. Therefore $\kappa \geqslant \mathfrak{b}$. □

Corollary 4.2. *If Y is a perfectly normal $\mathfrak{b}$-Luzin space with a countable base of open sets, then Y is not an* mQN-*space.*

Proof. Y is metrizable and hence Y is a subspace of a Polish space P. There is a countable dense set A in $\mathrm{cl}(Y)$ disjoint from Y. $Y \cup A$ is $\mathfrak{b}$-concentrated on A and therefore we can apply Theorem 4.1 for $X = Y \cup A$ and Y. □

Theorem 4.3. *Let X and Y be perfectly normal spaces with countable bases of open sets. If X is a $\mathfrak{b}$-Luzin space and f is a continuous function from X onto Y, then either $|Y| < \mathfrak{b}$ or Y is not an* mQN-*space.*

Proof. Let us assume that $|f(X)| \geqslant \mathfrak{b}$. Without loss of generality we assume that X, Y are subspaces of the Polish space $[0,1]^{\omega}$. Since subspaces of a κ-Luzin space are κ-Luzin spaces, without loss of generality we can assume that f is one-to-one. Let $U \subseteq X$ be the largest countable open set in X. Since the property mQN is countably additive it is enough to prove the theorem for $\mathfrak{b}$-Luzin space $X \setminus U$ instead of X and without loss of generality let us assume that $X = X \setminus U$. There is a Borel set $G \subseteq [0,1]^\omega$ and a continuous function $\tilde{f}: G \to [0,1]^\omega$ such that $X \subseteq G \subseteq \mathrm{cl}(X)$ and $\tilde{f}$ extends f (find sets G_n and extensions $\tilde{f}_n : G_n \to \mathbb{R}$ for coordinates f_n of the function f and set $G = \bigcap_{n=0}^{\infty} G_n$). For every open set $V \subseteq [0,1]^\omega$ if $G \cap V \neq \emptyset$, then $\tilde{f}(G \cap V)$ is uncountable and hence the uncountable analytic set $\tilde{f}(G \cap V)$ contains C a copy of Cantor set. But $f(X)$ is $\mathfrak{b}$-concentrated on a countable set and hence it cannot contain C. Therefore $\tilde{f}(G \cap V) \setminus f(X) \neq \emptyset$. Consequently, we can find a countable dense set $A \subseteq G$ such that $\tilde{f}(A) \cap f(X) = \emptyset$. The $\mathfrak{b}$-Luzin space $X \cup A$ is $\mathfrak{b}$-concentrated on $\tilde{f}(A)$. Since $f(X) \subseteq \tilde{f}(X \cup A) \setminus \tilde{f}(A)$, by Theorem 4.1 $f(X)$ is not an mQN-space. □

Corollary 4.4. *In the Cohen extension no totally imperfect set of reals of cardinality the continuum is an* mQN-*set.*

Proof. By a result of Miller [22], in the Cohen extension every totally imperfect set of reals X of cardinality the continuum is a continuous image of a Luzin set. By Theorem 4.3 X is not an mQN-set. □

Corollary 4.5. *No perfectly normal* mQN-*space with a countable base is a one-to-one continuous image of a perfectly normal* $\mathfrak{b}$-*Luzin space with a countable base.*

Problem 4.6. Is it consistent with ZFC that there exists a perfectly normal mQN-space of size $\geqslant \mathfrak{b}$, which is a continuous image of a $\mathfrak{b}$-Luzin set of reals?

5. $\overline{\mathrm{QN}}$-space

We will need the examples of sets the next lemma deal with. Let us set

$$K_0 = \left\{ x \in {}^\omega\omega \colon \sum_{n=0}^{\infty} 1/x(n) < \infty \text{ and } (\forall n)\ x(n) \leqslant 2^n \right\}.$$

The set K_0 is a non-empty σ-compact subset of the Baire space ${}^\omega\omega$.

Lemma 5.1. ${}^\omega 2$ *is not a* Σ-*space, and* K_0 *is not a* wΣQN-*space.*

Proof. (a) Let us introduce the following sequence of functions $f_n \in \mathcal{C}({}^\omega 2)$, $n \in \omega$ converging to 0,

$$f_n(x) = \begin{cases} 1/|\{i \leqslant n\colon x(i) = 1\}|, & \text{if } x(n) = 1, \\ 1/(n+1), & \text{if } x(n) = 0. \end{cases}$$

We prove that no subsequence of this sequence is Σ-convergent on ${}^\omega 2$. Let $\{n_k\}_{k=0}^{\infty}$ be an increasing sequence of integers. Let us take $x \in {}^\omega 2$ defined by $x(n) = 1$ if and only if $n = n_k$ for some k. Then $f_{n_k}(x) = 1/(k+1)$.

(b) Let $f_n \in \mathcal{C}(K_0)$, $n \in \omega$ be defined by $f_n(x) = 1/x(n)$. Then $\sum_{n=0}^{\infty} f_n(x) < \infty$ for $x \in K_0$. Let us assume that K_0 is a wΣQN-space and let $\sum_{m=0}^{\infty} 1/i_m < \infty$ be a given convergent series. There is an increasing sequence of integers $\{n_k\}_{k=0}^{\infty}$ such that $(\forall x \in K_0)$ $(\forall^\infty k)$ $f_{n_k}(x) < 1/k$. Let us define $x \in K_0$ by

$$x(n) = \begin{cases} i_m, & \text{if } n = n_{i_m}, \\ 2^n, & \text{if } n \notin \{n_{i_m}\colon m \in \omega\}. \end{cases}$$

Then $f_{n_{i_m}}(x) = 1/i_m$ for all m. This is a contradiction. □

Theorem 5.2. *The interval* $[0,1]$ *is neither an* $\overline{\mathrm{m}}$QN-*set, nor a* Σ-*set, nor a hereditary* mQN-*set, nor a* wΣQN-*set.*

Proof. The set ${}^\omega\omega$ of irrational numbers of the interval $[0,1]$ is not an mQN-set and so the interval is not a hereditary mQN-set. The space ${}^\omega 2$ is homeomorphic to a closed subset of the interval $[0,1]$ and the set K_0 is isomorphic to some σ-compact, and hence F_σ subset of the set of irrational numbers of the interval $[0,1]$. Hence by Lemma 5.1 the interval $[0,1]$ contains F_σ subsets which are not Σ-sets, and wΣQN-sets (an $\overline{\mathrm{m}}$QN-set is a wΣQN-set). By Theorem 2.3 we are done. □

Lemma 5.3. *Let X be either an* $\overline{\mathrm{m}}$QN-*space, or a hereditary* mQN-*space, or a* Σ-*space, or a* wΣQN-*space.*

(1) *If X is completely regular then X has a clopen basis.*

(2) *If X is a perfectly normal space then every open set in X is a countable union of clopen sets.*

Proof. By Theorem 5.2 every continuous image of X into the reals is totally disconnected.

(1) Let $x \in U$ and U be an open set. There is $f \in \mathcal{C}(X)$ such that $f(x) = 1$ and $f(y) = 0$ for all $y \in X \setminus U$. The image $f(X)$ does not contain an interval, hence there is a relatively clopen set $V \subseteq f(X)$ containing 1. Then $x \in f^{-1}(V) \subseteq U$ and $f^{-1}(V)$ is clopen in X.

(2) Let $U \subseteq X$ be an open set. X is perfectly normal, and so $X \setminus U$ is a zero set of a function $f \in \mathcal{C}(X)$. Now, the set $f(X) \setminus \{0\}$ is a countable union of relatively clopen sets in $f(X)$ and so also $U = f^{-1}(f(X) \setminus \{0\})$ is a such set. □

Theorem 5.4. *Let X be a perfectly normal space. If X is either a hereditary* mQN-*space, or an* $\overline{\mathrm{m}}$QN-*space, or a* Σ-*space, then* $\mathrm{Open}(X)$ *is weakly distributive.*

Proof. Notice that in all cases X is an mQN-space (Σ-space is an mQN-space by Lemma 1.8). Hence the assertion follows from Lemma 5.3 and Theorem 3.9. □

Theorem 5.5. *Let Y be either a Polish space or a perfectly normal hereditary separable σ-compact space. If $X \subseteq Y$ is a hereditary* mQN-*space, or an* $\overline{\mathrm{m}}$QN-*space, or a* Σ-*space, then X is perfectly meager.*

Proof. Uncountable compact sets can be mapped onto the interval $[0,1]$, therefore by Theorem 5.2 X has no uncountable compact subset. By assertions (5) and (6) of Theorem 3.8 X is perfectly meager. □

Problem 5.6. Is there a perfectly normal Σ-space which is not perfectly meager?

Recław [24] proved that every metric QN-space is a σ-set. Now we improve this result for perfectly normal ΣQN-spaces.

Theorem 5.7. *If X is a perfectly normal ΣQN-space, then X is a σ-space.*

Proof. (1) Let $w_n : \mathbb{R} \to [0,1]$ be continuous functions such that $w_n(x) = 1$ for $x \in [2^{-n-1}, 2^{-n}]$, and $w_n(x) = 0$ for $x \leqslant 2^{-n-2}$ and for $x \geqslant 2^{-n+1}$. Note that $\sum_{n=0}^{\infty} w_n(x) \leqslant 3$. Let $G = \bigcap_{k=0}^{\infty} U_k$ with U_k open. For each $k \in \omega$ there is a continuous function $g_k : X \to [0,1]$ such that $X \setminus U_k = g_k^{-1}(\{0\})$. Let us define $f_n \in \mathcal{C}(X)$ by

$$f_n(x) = \sum_{k=0}^{\infty} 2^{-k} w_n\big(g_k(x)\big).$$

Then

$$\sum_{n=0}^{\infty} f_n(x) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} 2^{-k} w_n\big(g_k(x)\big) \leqslant \sum_{k=0}^{\infty} 3 \cdot 2^{-k} < \infty.$$

As X is a ΣQN-space, there is a sequence of positive reals $\{\varepsilon_n\}_{n=0}^{\infty}$ such that $\lim_{n\to\infty} \varepsilon_n = 0$ and $(\forall x \in X)(\forall^{\infty} n)\ f_n(x) \leqslant \varepsilon_n$. Hence $X = \bigcup_{m=0}^{\infty} X_m$, where $X_m = \{x \in X\colon (\forall n \geqslant m)\ f_n(x) \leqslant \varepsilon_n\}$ are closed. We show that $X_m \cap U_k$ are closed. There is $n_0 \geqslant m$ such that $\varepsilon_n < 2^{-k}$ for all $n \geqslant n_0$. For $x \in X_m \cap U_k$ we have $2^{-k} w_n(g_k(x)) \leqslant f_n(x) \leqslant \varepsilon_n < 2^{-k}$ and so $w_n(g_k(x)) < 1$ for all $n \geqslant n_0$. Consequently, for every $n \geqslant n_0$ we have $g_k(x) \notin [2^{-n-1}, 2^{-n}]$. Since $g_k(x) > 0$ we obtain $g_k(x) \geqslant 2^{-n_0}$. Therefore $X_m \cap U_k = \{x \in X_m\colon g_k(x) \geqslant 2^{-n_0}\}$. This proves that the set $G = \bigcup_{m=0}^{\infty}(\bigcap_{k=0}^{\infty} X_m \cap U_k)$ is an F_σ set. □

Theorem 5.8. *If X is a perfectly normal ΣQN-space, then* $\mathrm{Open}(X)$ *is weakly distributive.*

Proof. By Theorems 2.3, 3.9, and Lemma 5.3(2) without loss of generality we can assume that $X \subseteq {}^{\omega}\omega$. It is enough to prove that X is bounded. Let $f_{n,m} : {}^{\omega}\omega \to \mathbb{R}$ be continuous functions defined by

$$f_{n,m}(x) = \begin{cases} 2^{-n}, & \text{if } x(n) = m, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\sum_{n,m=0}^{\infty} f_{n,m}(x) = \sum_{n=0}^{\infty} f_{n,x(n)}(x) < \infty$ for all $x \in {}^{\omega}\omega$. As X is a ΣQN-space, there is a sequence of positive reals $\{\varepsilon_{n,m}\}_{n,m=0}^{\infty}$ converging to 0 such that $(\forall x \in X)$ $(\forall^{\infty}(n,m) \in \omega \times \omega)\ f_{n,m}(x) \leqslant \varepsilon_{n,m}$, and hence $(\forall x \in X)(\forall^{\infty} n)\ 2^{-n} \leqslant \varepsilon_{n,x(n)}$. Let $\varphi(n) = \min\{m\colon (\forall k > m)\ \varepsilon_{n,k} < 2^{-n}\}$. Clearly, φ eventually dominates all members of X. □

Similarly as in the case of mQN-sets (Theorem 3.12) most of the properties studied in this paper are hereditary in σ-sets (the notions of ΣQN-sets, QN-sets, $\overline{\mathrm{QN}}$-sets, $\Sigma\Sigma^*$-sets,

$\overline{\Sigma}$QN-sets, etc. are hereditary, but the notions of mQN-sets, wQN-sets, and Σ-sets are not hereditary, see [6]).

Problem 5.9.
(1) Is there a perfectly normal hereditary mQN-space which is not a ΣQN-space?
(2) Is there a perfectly normal hereditary Σ-space (wQN-space) which is not a QN-space?

The equivalence of conditions (2) and (4) in the next theorem has been proved by Bartoszyński and Scheepers [2].

Theorem 5.10. *Let X be a perfectly normal space. The following conditions are equivalent.*
(1) Closed(X) *is weakly distributive.*
(2) Borel(X) *is weakly distributive.*
(3) *For every F_σ measurable function $f : X \to {}^\omega\omega$ the image $f(X)$ is bounded.*
(4) *For every Borel measurable function $f : X \to {}^\omega\omega$ the image $f(X)$ is bounded.*
(5) *X is an $\mathcal{M}_1$*QN-*space.*
(6) *X is an $\mathcal{M}$*QN-*space.*
(7) *X is a* $\overline{\text{QN}}$-*space.*
(8) *X is an* $\overline{\text{m}}$QN-*space.*
(9) *$\mathcal{D}_1(X) = \mathcal{A}m_1(X)$ and X is a* QN-*space.*
(10) *$\mathcal{D}_1(X) = \mathcal{M}_1(X)$ and X is a* wQN-*space.*
(11) *$\mathcal{D}_1(X) = \mathcal{M}_1(X)$ and X is an* mQN-*space.*

Proof. The equivalence (1) $\equiv$ (2) is proved in [6, Corollary 5.3]. The equivalences (1) $\equiv$ (3) and (2) $\equiv$ (4) and the implication (2) $\Rightarrow$ (6) hold true by Theorem 2.1. Notice that X is a σ-space if and only if $\mathcal{A}m_1(X) = \mathcal{M}_1(X)$. Actually, if $G = \bigcap_{n=0}^\infty U_n$ is a G_δ set, with $X = U_0 \supseteq U_1 \supseteq \cdots$, then $G = f^{-1}(\{1\})$ for the function $f \in \mathcal{M}_1(X)$ defined by $f(x) = \sum_{x\in U_n} 2^{-n-1}$. Therefore the implication (9) $\Rightarrow$ (10) follows from Theorem 5.7. The implications (6) $\Rightarrow$ (5) $\Rightarrow$ (7) $\Rightarrow$ (9), (10) $\Rightarrow$ (11), and (5) $\Rightarrow$ (8) are trivial. The implication (11) $\Rightarrow$ (3) is a consequence of Theorem 2.3 and Lemma 3.2 since every function $f \in \mathcal{M}_1(X, {}^\omega\omega)$ can be treated as an element of $\mathcal{M}_1(X) = \mathcal{D}_1(X)$ and so $f \in \mathcal{D}_1(X, {}^\omega\omega)$.

We prove the implication (8) $\Rightarrow$ (1). Let $X = \bigcap_{n=0}^\infty \bigcup_{m=0}^\infty F_{n,m}$, with $F_{n,m}$ closed. By Lemma 5.3 $F_{n,m} = \bigcup_{k=0}^\infty U_{n,m,k}$, with $U_{n,m,k}$ clopen and $U_{n,m,k+1} \subseteq U_{n,m,k}$ for all k. Let us define the functions $g, g_k : X \to {}^\omega\omega$ by

$$g_k(x)(2n) = \min\{m \in \omega\colon\ x \in U_{n,m,k}\},$$

$$g(x)(2n) = \min\{m \in \omega\colon\ x \in F_{n,m}\},$$

$$g_k(x)(2n+1) = g(x)(2n+1) = 0.$$

If we identify ${}^\omega\omega$ with the set of irrational numbers of the interval $[0, 1]$ via continued fractions we can see that in fact $g_{k+1} \leqslant g_k$ and g_k converge to g. Therefore the sequence of continuous functions g_k, $k \in \omega$ converges quasi-normally on X and so $g \in \mathcal{D}_1(X)$. Let $f_n \in \mathcal{C}(X)$, $n \in \omega$ be such that $(\forall x \in X)(\forall^\infty n)\ f_n(x) = g(x)$. The closed sets $A_k = \{x \in X\colon (\forall n \geqslant k)\ f_n(x) = f_k(x)\}$ for $k \in \omega$ cover X and the sets $g(A_k) = f_k(A_k)$ are all mQN-subsets of ${}^\omega\omega$ (Theorem 2.3). Hence also the set $g(X) = \bigcup_{k=0}^\infty f_k(A_k)$ is an mQN-set and by Lemma 3.2 it is a bounded subset of ${}^\omega\omega$. Let $\varphi \in {}^\omega\omega$ be such that $(\forall x \in X)(\forall^\infty n)$ $g(x)(n) \leqslant \varphi(n)$. This means that $X \subseteq \bigcup_{k=0}^\infty \bigcap_{n=k}^\infty \bigcup_{m\leqslant\varphi(n)} F_{n,m}$. □

As an addition to the theorem let us remark that in [25] it is proved that a perfectly normal space X is a $\overline{\mathrm{QN}}$-space if and only if for every monotone sequence of continuous functions $\{f_n\}_{n=0}^\infty$ such that $\sum_{n=0}^\infty |f_n(x)| < \infty$ on X there exists a monotone unbounded sequence of integers $\{k_n\}_{n=0}^\infty$ such that $\sum_{n=0}^\infty k_n|f_n(x)| < \infty$ on X (the word "monotone" can be excluded and "continuous" replaced by "Borel" in this equivalence).

6. Σ-convergence and Σ^*-convergence

Theorem 6.1. *Let X be a perfectly normal space. X is an $\mathrm{m}\Sigma\Sigma^*$-space if and only if X is a $\Sigma\Sigma^*$-space.*

Proof. By Theorem 5.4 and Lemma 3.3 every perfectly normal $\mathrm{m}\Sigma\Sigma^*$-space is nestled. Let X be an $\mathrm{m}\Sigma\Sigma^*$-space. Let $\sum_{n=0}^\infty |f_n(x)| < \infty$ for $x \in X$. We want to obtain the Σ^*-convergence. Without loss of generality we can assume that $f_n(x) > 0$ (otherwise take $f_n'(x) = \max\{f_n(x), 2^{-n}\}$). By Lemma 3.1 there are integers m_n such that $(\forall x \in X)(\forall^\infty n)$ $f_{n+1}(x)/m_{n+1} < f_n(x)/m_n$. The sets

$$F_k = \big\{x \in X\colon (\forall n \geqslant k)\ f_{n+1}(x)/m_{n+1} \leqslant f_n(x)/m_n\big\}$$

are closed, by Theorem 2.3 they are $\mathrm{m}\Sigma\Sigma^*$-spaces, and $X = \bigcup_{k=0}^\infty F_k$. The monotone series $\sum_{n=k}^\infty \sum_{i=1}^{m_n} f_n/m_n$ converges on F_k and hence it Σ^*-converges on F_k. Therefore the series $\sum_{n=0}^\infty f_n(x)$ Σ^*-converges on F_k for every $k \in \omega$ and hence the series Σ^*-converges on the whole X (since every countable sequence of convergent series can be majorized by a single convergent series). □

Theorem 6.2. *Let $\mathcal{F}$ be any class of Borel functions containing all continuous functions. For perfectly normal spaces the following equivalences between the properties hold true*:

(i) $\overline{\mathrm{QN}} \equiv \overline{\mathcal{F}}\mathrm{QN} \equiv \mathrm{w}\overline{\mathcal{F}}\mathrm{QN} \equiv \overline{\mathcal{F}}^{\downarrow}\mathcal{QN}$,

(ii) $\Sigma\Sigma^* \equiv \mathcal{F}\Sigma\Sigma^* \equiv \mathcal{F}^{\downarrow}\Sigma\Sigma^* \equiv \overline{\mathcal{F}}\Sigma\Sigma^* \equiv \overline{\mathcal{F}}^{\downarrow}\Sigma\Sigma^*$,

(iii) $\mathcal{M}\Sigma\mathrm{QN} \equiv \mathcal{M}_1\Sigma\mathrm{QN}$.

Proof. We can assume that $\mathcal{F}$ satisfies conditions (F1)–(F4).

(i) By Diagram 1

$$\mathcal{M}\mathrm{QN} \to \overline{\mathcal{F}}\mathrm{QN} \to \mathrm{w}\overline{\mathcal{F}}\mathrm{QN} \to \overline{\mathcal{F}}^{\downarrow}\mathrm{QN} \to \overline{\mathrm{QN}} \to \overline{\mathrm{m}}\mathrm{QN}$$

and by equivalence (6) ≡ (8) of Theorem 5.10 the equivalences hold true.

(ii) $\mathcal{F}\Sigma\Sigma^* \equiv \overline{\mathcal{F}}\Sigma\Sigma^* \to \overline{\mathcal{F}}^{\downarrow}\Sigma\Sigma^* \to \mathcal{F}^{\downarrow}\Sigma\Sigma^* \to \mathrm{m}\Sigma\Sigma^* \equiv \Sigma\Sigma^*$ and $\Sigma\Sigma^* \to \Sigma\mathrm{QN}$ by Lemma 1.7 and Theorem 6.1. Therefore it is enough to prove that $\Sigma\Sigma^* \to \mathcal{F}^{\downarrow}\Sigma\Sigma^*$. Let X be a perfectly normal $\Sigma\Sigma^*$-space and let $\sum_{n=0}^{\infty}|f_n(x)| < \infty$ with $f_n \in \mathcal{F}(X)$. Since X is a σ-space (Theorem 5.7), $f_n \in \mathcal{M}_1(X)$ and so there are $f_{n,m} \in \mathcal{C}(X)$ for $m \in \omega$ such that $f_n(x) = \lim_{n\to\infty} f_{n,m}(x)$ on X. Moreover, there are $\varepsilon_{n,m} > 0$ with $\varepsilon_{n,m+1} \leqslant \varepsilon_{n,m}$ and $\lim_{m\to\infty}\varepsilon_{n,m} = 0$ such that $(\forall x \in X)(\forall^{\infty} m)\ |f_n(x) - f_{n,m}(x)| < \varepsilon_{n,m}$ for all $n \in \omega$. The function $g : X \to {}^{\omega}\omega$ defined by

$$g(x)(n) = \min\{m\colon\ (\forall k \geqslant m)\ |f_n(x) - f_{n,k}(x)| < \varepsilon_{n,k}\}$$

is Borel and so there is $\varphi \in {}^{\omega}\omega$ such that $(\forall x \in X)(\forall^{\infty} n)\ |f_n(x) - f_{n,\varphi(n)}(x)| < \varepsilon_{n,\varphi(n)}$. We can choose φ so that $\sum_{n=0}^{\infty}\varepsilon_{n,\varphi(n)} < \infty$. Now $\sum_{n=0}^{\infty}|f_{n,\varphi(n)}(x)| < \infty$ and since X is a $\Sigma\Sigma^*$-space there is a positive series $\sum_{n=0}^{\infty}\varepsilon'_n < \infty$ such $(\forall x \in X)(\forall^{\infty} n)\ |f_{n,\varphi(n)}(x)| < \varepsilon'_n$. Then $(\forall x \in X)(\forall^{\infty} n)\ |f_n(x)| \leqslant \varepsilon_{n,\varphi(n)} + \varepsilon'_n$ and $\sum_{n=0}^{\infty}(\varepsilon_{n,\varphi(n)} + \varepsilon'_n) < \infty$.

(iii) Every $\mathcal{M}_1\Sigma\mathrm{QN}$-space X is a σ-space and hence $\mathcal{M}(X) = \mathcal{M}_1(X)$. □

Lemma 6.3. *If X is a separable metric* wΣQN*-space without isolated points, then X is meager.*

Proof. The proof is a modification of the proof of the same Theorem 4.2 in [6] for wQN-spaces.

Let $\{r_n\colon\ n \in \omega\}$ be a countable dense subset of X. For every $n \in \omega$ let $x_{n,m} \in X$ be such that $\rho(r_n, x_{n,m}) \geqslant 2\rho(r_n, x_{n,m+1})$ for each $n \in \omega$. Let $f_{n,m} : X \to [0,1]$ be a continuous function, $f_{n,m}(x_{n,m}) = 1$ and $f_{n,m}(x) = 0$ for $\rho(x, x_{n,m}) \geqslant \rho(r_n, x_{n,m})/4$. The functions $h_m(x) = \sum_{n=0}^{\infty} 2^{-n} f_{n,m}(x)$, $m \in \omega$ are continuous and $\sum_{m=0}^{\infty} h_m(x) = \sum_{n=0}^{\infty} 2^{-n}\sum_{m=0}^{\infty} f_{n,m}(x) \leqslant \sum_{n=0}^{\infty} 2^{-n} < \infty$, because if $m_1 \neq m_2$, then $f_{n,m_1}(x) = 0$ or $f_{n,m_2}(x) = 0$ for every $x \in X$. Since X is a wΣQN-space, X is the union of closed sets X_i, $i \in \omega$ such that some subsequence $\{h_{m_k}\}_{k=0}^{\infty}$ converges uniformly to 0 on each set X_i. If X is not meager, then there is i such that $\mathrm{Int}(X_i) \neq \emptyset$. Let us fix $r_k \in \mathrm{Int}(X_i)$. Then $x_{n,m} \in X_i$ for $m \geqslant m^*$ and $h_{m_k}(x_{n,m_k}) \geqslant 2^{-n}$ for $m_k \geqslant m^*$. This contradicts the uniform convergence on X_i. □

Theorem 6.4. *If X is a perfectly normal* wΣQN*-space with a countable base of open sets, then X is perfectly meager.*

Proof. A perfect subset P of X is a wΣQN-space and having no isolated points by Lemma 6.3 it is meager in itself. □

If X is a hereditary separable σ-space, then X is perfectly meager (see [23, Theorem 5.2]). This gives another sufficient condition (in addition to Theorems 5.5 and 6.4) for a ΣQN-space to be perfectly meager.

Problem 6.5. Is there a perfectly normal wΣQN-space (ΣQN-space) which is not perfectly meager?

We say that X is an m-*space* if for every sequence of strictly positive functions $f_n \in \mathcal{C}(X)$, $n \in \omega$ converging to zero on X there is an increasing sequence of integers n_k such that $(\forall x \in X)(\forall^\infty k)\ f_{n_{k+1}}(x) \leqslant f_{n_k}(x)$. A space X is an *M-space* if for every sequence of functions $f_n \in \mathcal{C}(X)$, $n \in \omega$ such that $\sum_{n=0}^{\infty} |f_n(x)| < \infty$ for $x \in X$ there exists an increasing sequence of integers $\{n_k\}_{k=0}^{\infty}$ such that for every $x \in X$ $\sum_{n=n_k}^{n_{k+1}-1} |f_n(x)| \leqslant \sum_{n=n_{k-1}}^{n_k-1} |f_n(x)|$ for all but finitely many k.

Theorem 6.6. *If X is a perfectly normal* wQN-*space, then X is an* m-*space.*

Proof. Open(X) is weakly distributive by Theorem 5.4 (or by [6, Theorem 5.8]). Hence by Lemma 3.3(2) there is a sequence of integers k_n such that $(\forall x \in X)(\forall^\infty n)\ f_n(x) > 1/k_n$. Since X is a wQN-space there is an increasing sequence of integers m_n such that $(\forall x \in X)$ $(\forall^\infty n)\ f_{m_n}(x) \leqslant 1/k_n$. Now set $n_0 = 0$ and $n_{k+1} = m_{n_k}$. □

By Lemma 1.2 every m-space is a wΣQN-space and by Theorem 6.6 we can easily see that a space X is a wQN-space if and only if X is an m-space and an mQN-space.

Theorem 6.7. non(m-*space*) $= \mathfrak{b}$.

Proof. Let $\{g_\alpha\colon \alpha < \mathfrak{b}\} \subseteq {}^\omega\omega$ be an unbounded family of strictly increasing functions. Let $X = \{x_\alpha\colon \alpha < \mathfrak{b}\}$ be the subset of ${}^\omega\omega$ defined by $x_\alpha(n) = m + g_\alpha(m+1) - n$ for $n \in [g_\alpha(m), g_\alpha(m+1))$. Let $f_n : X \to \mathbb{R}$ be defined by $f_n(x) = 1/x(n)$. The set X is not an m-space because if $\{n_k\}_{k=0}^{\infty}$ is an increasing sequence of integers, then for some $\alpha < \mathfrak{b}$ for infinitely many m there is k such that $g_\alpha(m) \leqslant n_k < n_{k+1} < g_\alpha(m+1)$ and hence $f_{n_{k+1}}(x_\alpha) > f_{n_k}(x_\alpha)$ for infinitely many k. We have proved that the minimal size of a space which is not an m-space is $\leqslant \mathfrak{b}$. By Theorem 6.6 this cardinal is also $\geqslant \mathfrak{b}$. □

Problem 6.8.

(1) Is there a Σ-space which is not an m-space (and hence $\Sigma \neq \mathrm{w}\mathcal{QN}$)?
(2) Is there an m-space which is not a wQN-space?

Theorem 6.9. *Every M-space is a* ΣQN-*space and every perfectly normal* $\overline{\text{QN}}$-*space is an M-space.*

Proof. The first part is a consequence of Lemma 1.2. If X is a perfectly normal $\overline{\text{QN}}$-space and $\sum_{n=0}^{\infty} |f_n(x)| < \infty$ for $x \in X$, then the mapping $\varphi : X \to {}^\omega\omega$ defined by

$$\varphi(x)(k) = \min\left\{m > k\colon \sum_{n=m}^{\infty} \left|f_n(x)\right| \leqslant \sum_{n=k}^{m-1} \left|f_n(x)\right|\right\}$$

is Borel and by Theorem 5.10 the image $\varphi(X)$ is bounded. Let $\psi \in {}^\omega\omega$ be a bound for $\varphi(X)$. Then the sequence n_0, $n_{k+1} = \psi(n_k)$ is a witness for the property M. □

Problem 6.10. Is it consistent with ZFC that there is a QN-space which is not an M-space?

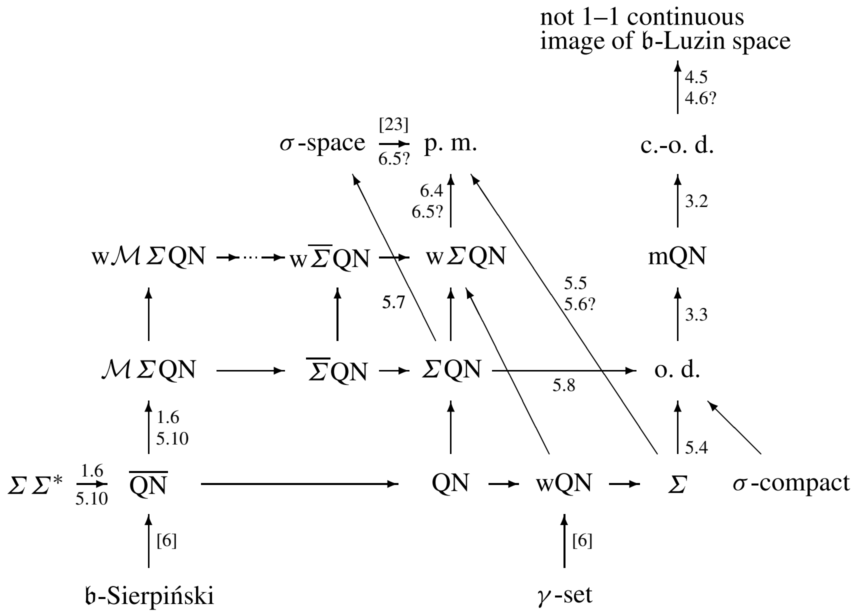

Diagram 2. The implications in the diagram hold true for perfectly normal spaces with some exceptions marked by question marks at the numbers of references of corresponding problems. No reference at an arrow means that the implication is an easy consequence of definitions. The abbreviations: p. m. (perfectly meager), o. d. (open distributivity), and c.-o. d. (clopen distributivity). Some equivalences (for perfectly normal spaces):

(1) $\Sigma\Sigma^* \equiv \mathrm{m}\Sigma\Sigma^* \equiv \mathcal{M}\Sigma\Sigma^* \equiv \mathcal{M}^{\downarrow}\Sigma\Sigma^*$ (6.2),
(2) $\overline{\mathrm{QN}} \equiv \mathcal{M}\mathrm{QN} \equiv \overline{\mathrm{m}}\mathrm{QN} \equiv \mathcal{M}^{\downarrow}\mathrm{QN} \equiv$ closed distributivity $\equiv$ Borel distributivity (5.10, 6.2),
(3) $\mathcal{M}\Sigma\mathrm{QN} \equiv \mathcal{M}_1\Sigma\mathrm{QN}$ (5.7),
(4) $\overline{\Sigma}\mathrm{QN} \equiv \Sigma\mathrm{QN}$ & $\mathrm{w}\overline{\Sigma}\mathrm{QN}$ (2.4),
(5) $\mathrm{wQN} \equiv \Sigma$ & $\mathrm{w}\Sigma\mathrm{QN} \equiv \mathrm{m}$ & mQN (6.6).

Diagram 2 shows the relationships among considered properties, as established at this point.

Problem 6.11. Can in ZFC some other implications be proved between properties in Diagram 2 (for sets of reals, or a class of spaces)?

7. Several examples

The following result was proved by Recław [24] using the construction of an uncountable $S_1(\Gamma, \Gamma)$ set in [17] (in [17] it is proved only that this construction leads to a weaker property $S_1(\Gamma, \Gamma)^*$ and later Scheepers, in [28], proves that this property is equivalent to $S_1(\Gamma, \Gamma)$). In fact, he presents only the assertion (i), but the same proof works for the assertion (ii) and the case (i) is derived from the case (ii) by taking X to be any set of reals of size ω_1 provided that $\mathfrak{b} > \omega_1$.

Theorem 7.1.
(i) *There exists an uncountable* wQN*-set* $X \subseteq \mathbb{R}$.
(ii) *If* $\mathfrak{t} = \mathfrak{b}$, *then there exists a* wQN*-set* $X \subseteq \mathbb{R}$ *of size* $\mathfrak{b}$ *which is* $\mathfrak{b}$*-concentrated on a countable subset.*

This theorem immediately implies the existence of a wQN-set of size $\mathfrak{t}$ (see [27]). Let us note that Scheepers under the assumption $\mathfrak{b} = \mathfrak{t}$ constructs an $S_1(\Gamma, \Gamma)$-set of cardinality $\mathfrak{b}$ whose no subset of cardinality $\mathfrak{b}$ is a QN-set.

Every Sierpiński set is a $\overline{\text{QN}}$-set. Under CH there is a Sierpiński set $S \subseteq \mathbb{R}$ such that $S + S = \mathbb{R}$. Then, since $\mathbb{R}$ is a continuous image of $S \times S$, $S \times S$ is neither a Σ-space, nor a wΣQN-space, nor a $\Sigma\Sigma^*$-space. In particular none of these properties is preserved by finite powers.

Notice that if $S \subseteq \mathbb{R}$ is an uncountable (Sierpiński) set then $S \times S$ is not a Sierpiński set.

Theorem 7.2. (CH) *There is a Sierpiński set S such that* S^n *is a* $\overline{\text{QN}}$*-set for every* $n \geqslant 1$.

Proof. We prove the assertion for $n = 2$ only. The proof in general is analogous. Let $\{N_\alpha\colon \alpha < \omega_1\}$ and $\{M_\alpha\colon \alpha < \omega_1\}$ be Borel bases of the ideals of measure zero sets in $\mathbb{R}$ and in $\mathbb{R} \times \mathbb{R}$, respectively. Without loss of generality we can assume that both these systems are increasing with respect to the inclusion. By induction on $\alpha < \omega_1$ let us find $x_\alpha \in \mathbb{R}$ such that these two conditions are satisfied:
(1) $\mu((M_\alpha)_{x_\alpha}) = \mu((M_\alpha)^{x_\alpha}) = 0$,
(2) $x_\alpha \in \mathbb{R} \setminus \bigcup_{\beta<\alpha}((M_\beta)_{x_\beta} \cup (M_\beta)^{x_\beta} \cup N_\alpha)$.
Clearly the set $S = \{x_\alpha\colon \alpha < \omega_1\}$ is a Sierpiński set and if $M \subseteq \mathbb{R} \times \mathbb{R}$ has measure zero, then there is a countable set $A \subseteq S$ such that $(S \times S) \cap M \subseteq (A \times S) \cup (S \times A) \cup \{(x, x)\colon x \in S\}$.

Let $f_n : S \times S \to \mathbb{R}$, $n \in \omega$ be a sequence of Borel functions converging to 0. We can extend the functions f_n to Borel functions f'_n defined on $\mathbb{R} \times \mathbb{R}$. By Egoroff's theorem there is a set $H \subseteq \mathbb{R} \times \mathbb{R}$ such that $\mu((\mathbb{R} \times \mathbb{R}) \setminus H) = 0$ and f'_n QN-converges to 0. Let $\{\varepsilon'_n\}_{n=0}^{\infty}$ witness that. Let $A \subseteq S$ be a countable set such that $(S \times S) \setminus H \subseteq Z$, where $Z = (A \times S) \cup (S \times A) \cup \{(x, x)\colon x \in S\}$. Since Z is a $\overline{\text{QN}}$-set we can find $\{\varepsilon''_n\}_{n=0}^{\infty}$ witnessing the QN-convergence of f'_n to 0 on Z. Now the sequence $\varepsilon_n = \max\{\varepsilon'_n, \varepsilon''_n\}$, $n \in \omega$ witnesses the QN-convergence of f_n to 0 on $S \times S$. □

Todorcevic in [29, Propositions 6.0 and 6.1] proves that under Martin's Axiom (the additivity of Lebesgue measure is $\mathfrak{c}$, is enough to assume) there is a strongly Sierpiński set, i.e., a set $S = \{x_\alpha\colon \alpha < \mathfrak{c}\}$ or reals such that for every $n \geqslant 1$ and for every measure zero set $N \subseteq \mathbb{R}^n$ there is $\beta < 2^{\aleph_0}$ such that $S \cap N$ contains no k-tuple with all indexes above β. The referee noticed that the assertion of Theorem 7.2 holds true for every strongly Sierpiński set. This is true without CH assuming only $\mathfrak{b} = \mathfrak{c}$. The proof uses the same arguments as the proof of Theorem 7.2, the fact that the class $\overline{\mathcal{QN}}$ is $\mathfrak{b}$-additive, and is carried out by induction on $n \geqslant 1$.

The product of two different γ-sets need not be a γ-set (see [14]). The following lemma was first proved probably by Daniels [8, Lemma 9].

Lemma 7.3. *If X is a γ-set, then X^n is a γ-set for each $n \geqslant 1$.*

Proof. Let $\mathcal{A}$ be an open ω-cover of X^n. If $F \subseteq X$ is a finite set, then there is an open set $V \in \mathcal{A}$ such that $F^n \subseteq V$. Hence there is an open set U_F such that $F \subseteq U_F$ and $(U_F)^n \subseteq V$. The family $\mathcal{B} = \{U_F\colon\ F \in [X]^{<\omega}\}$ is an ω-cover of X. There are sets $U_m \in \mathcal{B}$ such that $X \subseteq \bigcup_{k=0}^{\infty} \bigcap_{m \geqslant k} U_m$ and then $X^n \subseteq \bigcup_{k=0}^{\infty} \bigcap_{m \geqslant k} (U_m)^n$ and for each m there is $V_m \in \mathcal{A}$ such that $(U_m)^n \subseteq V_m$. $\square$

Under CH there is a Sierpiński set which is a linear subspace of $\mathbb{R}$ over $\mathbb{Q}$ [12]. We have the following.

Theorem 7.4. *If $X \subseteq \mathbb{R}$ is a γ-set and $F \subseteq \mathbb{R}$ is a countable field then the linear subspace of $\mathbb{R}$ over F generated by X is a* wQN-*space.*

Proof. The set

$$\bigcup_{\{a_1,\ldots,a_n\} \in [F]^{<\omega}} a_1 X + \cdots + a_n X$$

is a countable union of continuous images of powers of γ-sets. $\square$

If $|X| < \mathfrak{b}$, then X is a $\overline{\text{QN}}$-space and hence $\mathcal{M}(X) = \mathcal{D}_1(X)$. The next result was obtained by Kholshchevnikova [18] under MA.

Theorem 7.5. *Let X be a perfectly normal space with a countable base. Then ${}^A\mathbb{R} = \mathcal{D}_1(A)$ for every set $A \in [X]^{<\mathfrak{p}}$.*

Proof. By Silver's result [21] every subset of a set A of size $< \mathfrak{p}$ is an F_σ subset of A. Therefore ${}^A\mathbb{R} = \mathcal{M}_1(A)$. Now, since $|A| < \mathfrak{p} \leqslant \mathfrak{b}$, A is a $\overline{\text{QN}}$-space and so $\mathcal{M}_1(A) = \mathcal{D}_1(A)$. $\square$

The following is a slight strengthening of a similar result in [6].

Theorem 7.6 ($\mathfrak{p} = \mathfrak{c}$). *There is a γ-set $X \subseteq \mathcal{P}(\omega)$ of size $\mathfrak{c}$ which is $\mathfrak{c}$-concentrated on the countable subset $[\omega]^{<\omega}$ and every Σ*QN-*subset of X has cardinality less than $\mathfrak{c}$.*

Proof. For $n < m$ let $f_{n,m} : \mathcal{P}(\omega) \to \mathbb{R}$ be defined by

$$f_{n,m}(x) = \begin{cases} 2^{-n}, & \text{if } x \cap [n,m] = \{n,m\}, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $\sum_{n<m<\omega} f_{n,m}(x) \leqslant \sum_{n \in x} 2^{-n} < \infty$. Let $y^* = \{z \subseteq \omega\colon\ z \subseteq^* y\}$. Using the same proofs as for Lemma 1.2 of [14] and for Lemma 6.6 of [6] we can prove this assertion:

Let $\mathcal{A}$ be an open ω-cover of $[\omega]^{<\omega}$, $x \in [\omega]^{\omega}$ and let $\{\varepsilon_{n,m}\}_{n,m}$ be a sequence of positive reals converging to 0. *There exist $D_n \in \mathcal{A}$ and $y \in [x]^{\omega}$ such that $y^* \subseteq \bigcup_m \bigcap_{n>m} D_n$ and for every $z \in y^*$ $f_{n,m}(z) > \varepsilon_{n,m}$ for infinitely many pairs (n, m).*

By induction we build $x_\alpha \in [\omega]^{\omega}$ such that $\alpha < \beta \to x_\beta \subseteq^* x_\alpha$ and we put $X = [\omega]^{<\omega} \cup \{x_\alpha\colon \alpha < \mathfrak{c}\}$. Let $\mathcal{A}_\alpha$, $\{\varepsilon^\alpha_{n,m}\}_{n,m}$ for $\alpha < \mathfrak{c}$ be all of the countable families of open sets and all of the sequences of positive reals converging to 0. If $\mathcal{A}_\alpha$ is an ω-cover of the countable set $X_\alpha = [\omega]^{<\omega} \cup \{x_\beta\colon \beta \leqslant \alpha\}$, then there exist $D_n \in \mathcal{A}_\alpha$ such that $X_\alpha \subset \bigcup_m \bigcap_{n>m} D_n$. Since $\{D_n\colon n \in \omega\}$ is again an ω-cover of $[\omega]^{<\omega}$ by the claim there exists $x_{\alpha+1} \in [x_\alpha]^{\omega}$ such that $x^*_{\alpha+1} \subseteq \bigcup_m \bigcap_{k>m} D_{n_k}$ for some n_k and every $z \in x^*_{\alpha+1}$ is a witness that $\{\varepsilon^\alpha_{n,m}\}_{n,m}$ is not a control sequence for QN-convergence of the sequence $\{f_{n,m}\}_{n,m}$. □

We give a characterization of $\Sigma\Sigma^*$-sets similar to the characterization of $\overline{\text{QN}}$-sets by condition (4) of Theorem 5.10. We set

$$L = \left\{x \in {}^{\omega}\omega\colon \sum_{n=0}^{\infty} x(n)/2^n \leqslant 1\right\}.$$

L is a closed subset of ${}^{\omega}\omega$.

Theorem 7.7. *The following conditions are equivalent.*

(i) *X is a $\Sigma\Sigma^*$-space.*

(ii) *Every continuous image of X into L is a bounded subset of L.*

(iii) *Every Borel image of X into L is a bounded subset of L.*

Proof. (i) ⇒ (iii) Let $f: X \to L$ be a Borel mapping. Since $\Sigma\Sigma^* = \mathcal{M}\Sigma\Sigma^*$ (Theorem 6.2(ii)), the space $X' = f(X)$ is again a $\Sigma\Sigma^*$-space. Let $f_n: L \to \mathbb{R}$ be defined by $f_n(x) = x(n)/2^n$. Then $\sum_{n=0}^{\infty} f_n(x) < \infty$ for $x \in X'$ and hence there is a convergent series $\sum_{n=0}^{\infty} \varepsilon_n$ such that $(\forall x \in X')(\forall^\infty n)\ x(n)/2^n \leqslant \varepsilon_n$. Let us define $y \in L$ by $y(n) = \min\{m\colon \varepsilon_n \leqslant m/2^n\}$. Then $(\forall x \in X')\ x \leqslant^* y$.

(ii) ⇒ (i) Let $f_n: X \to \mathbb{R}$, $n \in \omega$ be a Σ-convergent sequence of continuous functions. Since $f_n(X)$ is 0-dimensional, we can find reals $r_{n,m} \in \mathbb{R} \setminus f_n(X)$ such that $m/2^n < r_{n,m} < (m+1)/2^n$. Let $f: X \to L$ be the following continuous function: $f(x)(n) = \min\{m\colon f_n(x) < r_{n,m}\}$. There is $y \in L$ which eventually dominates all members of $f(X)$ and the sequence $\varepsilon_n = y(n)/2^n$, $n \in \omega$ witnesses Σ^*-convergence of f_n, $n \in \omega$. □

Theorem 7.8. *None of the implications $\Sigma\Sigma^* \to \gamma$, $\gamma \to \sigma$, $\sigma \to$ mQN is provable.*

Proof. (a) It is well known that $\mathfrak{b}(L, \leqslant^*) = \text{add}(\mathcal{N})$ and $\mathfrak{d}(L, \leqslant^*) = \text{cof}(\mathcal{N})$. Since the inequality $\mathfrak{p} < \text{add}(\mathcal{N})$ is consistent, also $\Sigma\Sigma^* \nsubseteq \gamma$ is consistent.

(b) By Theorem 7.6 under Martin's axiom there exists a γ-set of reals of size $\mathfrak{c}$ which is $\mathfrak{c}$-concentrated on a countable subset (see also [14,6]). By Theorems 3.12 and 4.1 this γ-set is not a σ-space.

(c) Miller [23, Theorem 5.7] under CH has constructed an uncountable σ-set X of reals concentrated on a countable set. By Theorems 4.1 and 3.12 X is not an mQN-set. □

Every hereditary γ-set is a σ-space. The next question could be interesting.

Problem 7.9. Is there a γ-set which is a σ-space but not a hereditary γ-set?

The minimal cardinalities of sets which do not have a property in Diagram 2 are equal either to add($\mathcal{N}$), or $\mathfrak{b}$, or the following cardinal invariants which lie between $\mathfrak{b}$ and non(Meager).

$$\mu = \min\bigl\{|\mathcal{F}|\colon\ \mathcal{F} \subseteq {}^{\omega}\omega \text{ and } (\forall h \in {}^{\omega}\omega)(\exists g \in \mathcal{F})(\exists^{\infty} n)\ h(n) \in \{g(m)\colon\ m < n\}\bigr\},$$
$$\mu' = \min\bigl\{|\mathcal{F}|\colon\ \mathcal{F} \subseteq {}^{\omega}\omega \text{ and } (\forall h \in {}^{\omega}\omega)(\forall A \in [\omega]^{\omega})(\exists g \in \mathcal{F})(\exists^{\infty} n \in g^{-1}(A))\ g(n) > h(n)\bigr\},$$
$$\mu'' = \min\bigl\{|\mathcal{F}|\colon\ \mathcal{F} \subseteq {}^{\omega}\omega \text{ and } \bigl(\forall \varphi,\ |\varphi(n)| \leqslant n\bigr)(\forall A \in [\omega]^{\omega})(\exists g \in \mathcal{F})(\exists^{\infty} n \in g^{-1}(A))\ g(n) \notin \varphi(n)\bigr\}.$$

Lemma 7.10. $\mu = \mu' = \mu''$.

Proof. The inequality $\mu'' \leqslant \mu'$ is trivial. Conversely, let $|\mathcal{F}| < \mu'$. There are h and A such that $(\forall g \in \mathcal{F})(\forall^{\infty} n \in g^{-1}(A))\ g(n) \leqslant h(n)$. We can assume that h is strictly increasing and $\operatorname{rng}(h) \subseteq A$. Let $A' = \operatorname{rng}(h) \setminus \{h(0)\}$ and $\varphi(n) = \{h(m)\colon\ 1 \leqslant m \leqslant n\}$. Then $|\varphi(n)| = n$ and $(\forall g \in \mathcal{F})(\forall^{\infty} n \in g^{-1}(A'))\ g(n) \in \varphi(n)$. Therefore $\mu' \leqslant \mu''$. Now, if $|\mathcal{F}| < \mu'$, then there exists an increasing function $h \in {}^{\omega}\omega$ such that for the set $A = \operatorname{rng}(h)$ for every $g \in \mathcal{F}$ we have $(\forall^{\infty} m \in g^{-1}(A))\ g(m) \leqslant h(m)$ and hence $(\forall^{\infty}(n, m))\ g(m) = h(n) \Rightarrow g(m) \leqslant h(m)$. As h is increasing $(\forall^{\infty} n)(\forall m \leqslant n)\ h(n) \neq g(m)$ holds true for every $g \in \mathcal{F}$. Hence $\mu' \leqslant \mu$.

We prove $\mu \leqslant \mu'$. Let $\mathcal{F} \subseteq {}^{\omega}\omega$, $|\mathcal{F}| = \mu'$ and $(\forall h \in {}^{\omega}\omega)(\forall A \in [\omega]^{\omega})(\exists g \in \mathcal{F})(\exists^{\infty} n \in g^{-1}(A))\ g(n) > h(n)$. We can assume that the function $g_0(n) = n$ is in $\mathcal{F}$. Let $h \in {}^{\omega}\omega$. If $\operatorname{rng}(h)$ is finite, then $(\exists^{\infty} n)\ h(n) \in \{g_0(m)\colon\ m < n\}$. Let $\operatorname{rng}(h)$ be infinite and let $\{k_n\}_{n=0}^{\infty}$ be an increasing sequence of integers such that $(\forall i < k_n)\ h(i) < h(k_n)$. Let us define $h^*(n) = h(k_n)$ and let $A = \operatorname{rng}(h^*)$. There is $g \in \mathcal{F}$ such that $h^*(n) < g(n)$ for infinitely many $n \in g^{-1}(A)$. Let $\{n_i\}_{i=0}^{\infty}$, $\{m_i\}_{i=0}^{\infty}$ be increasing such that $h^*(m_i) < g(m_i)$ and $g(m_i) = h^*(n_i)$. Then $m_i < n_i \leqslant k_{n_i}$ and $h(k_{n_i}) = h^*(n_i) \in \{g(m)\colon\ m < k_{n_i}\}$. Therefore $|\mathcal{F}| \geqslant \mu$. □

Theorem 7.11. $\operatorname{non}(\mathrm{w}\Sigma\mathcal{QN}) = \operatorname{non}(\mathrm{w}\mathcal{M}\Sigma\mathcal{QN}) = \mu$.

Proof. Let $\mathcal{F} \subseteq {}^{\omega}\omega$, $|\mathcal{F}| < \operatorname{non}(\mathrm{w}\Sigma\mathcal{QN})$. For $g \in {}^{\omega}\omega$ we define $x_g \in {}^{\omega}\omega$ by

$$x_g(n) = \begin{cases} \min\{i\colon\ n = g(i)\}, & \text{if } n \in \operatorname{rng}(g), \\ n, & \text{if } n \notin \operatorname{rng}(g). \end{cases}$$

The set $X = \{x_g\colon\ g \in \mathcal{F}\}$ is a wΣQN-set and since $\sum_{n=0}^{\infty} 2^{-x(n)} < \infty$ for $x \in X$, there are $A, B \in [\omega]^{\omega}$ such that $(\forall x \in X)(\forall^{\infty} k \in A)\ 2^{-x(k)} \leqslant 2^{-|B \cap k|}$. We can assume that $B \subseteq A$ and even $A = B$. Let $h : \omega \to A$ be the increasing enumeration of A. For $g \in \mathcal{F}$ for almost all k of the form $k = g(m) = h(n)$ we have $n = |A \cap k| \leqslant x_g(k) \leqslant m$. Therefore $(\forall^\infty n)$ $h(n) \notin \{g(m)\colon m < n\}$ and $\mathrm{non}(\mathrm{w}\Sigma\mathcal{QN}) \leqslant \mu$.

Let X be a non-w$\mathcal{M}\,\Sigma$QN-space and f_n be Borel functions on X such that $\sum_{n=0}^{\infty} f_n(x) < \infty$ and no subsequence of $\{f_n\}_{n=0}^{\infty}$ QN-converges. We can assume that f_n's are strictly positive. Let $n_x \in \omega$ be such that $\sum_{n=n_x}^{\infty} f_n(x) < 1$ and let us set $\varphi_x(n) = \{k \geqslant n_x\colon f_k(x) \geqslant 1/n\}$. Since $|\varphi_x(n)| < n$ and $\varphi_x(n) \subseteq \varphi_x(n+1)$ for each n we can find $g_x \in {}^{\omega}\omega$ such that $\varphi_x(n) \subseteq \{g_x(m)\colon m < n\}$. Let $h \in {}^{\omega}\omega$. There is $x \in X$ such that $f_{h(n)}(x) \geqslant 1/n$ and so either $h(n) < n_x$ or $h(n) \in \varphi_x(n) \subseteq \{g_x(m)\colon m < n\}$ for infinitely many n. The family $\mathcal{F} = \{g_x\colon x \in X\} \cup \{g_0\}$, where $g_0(n) = n$, witnesses $\mu \leqslant |X|$ and $\mu \leqslant \mathrm{non}(\mathrm{w}\mathcal{M}\,\Sigma\mathcal{QN})$. □

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