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TOWERS AND PERMITTED TRIGONOMETRIC THIN SETS

Abstract

In [3] we introduced the notion of perfect measure zero sets and proved that every perfect measure zero set is permitted for any of the four families of trigonometric thin sets \mathcal{N} , \mathcal{A} , \mathcal{N}_0 , and $p\mathcal{D}$. Now we prove that the unions of less than \aleph perfect measure zero sets are permitted for the mentioned families. This strengthens a result of T. Bartoszyński and M. Scheepers [1] saying that every set of cardinality less than \aleph is \mathcal{N} -permitted.

1 Introduction

Let \mathcal{F} be a family of sets of reals. Let $A, B \in \mathcal{F}$. We say that a set A is permitted for \mathcal{F} if $A \cup B \in \mathcal{F}$ for every $B \in \mathcal{F}$. Let A be a set of reals. Then A is a pD -set (pseudo Dirichlet set) if there is an increasing sequence of integers $\{n_k\}_{k=0}^{\infty}$ such that the sequence $\{\sin n_k \pi x\}_{k=0}^{\infty}$ converges quasi-normally on A , i.e. there is a sequence of positive reals $\{\varepsilon_k\}_{k=0}^{\infty}$ converging to 0 such that $(\forall x \in A)(\forall^{\infty} k) |\sin n_k \pi x| < \varepsilon_k$. A is an N_0 -set if there is an increasing sequence of integers $\{n_k\}_{k=0}^{\infty}$ such that $\sum_{k=0}^{\infty} |\sin n_k \pi x| < \infty$ for $x \in A$. A is an A -set if there is an increasing sequence of integers $\{n_k\}_{k=0}^{\infty}$ such that $\{\sin n_k \pi x\}_{k=0}^{\infty}$ converges to 0 for $x \in A$. A is an N -set if there is a sequence of non-negative reals $\{\rho_n\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty} \rho_n = \infty$ and the series $\sum_{n=0}^{\infty} |\rho_n \sin n \pi x|$ converges for $x \in A$. The families of all pD -sets, N_0 -sets, A -sets, and N -sets are denoted by $p\mathcal{D}$, \mathcal{N}_0 , \mathcal{A} , and \mathcal{N} , respectively.

A set A has perfect measure zero if for every sequence of positive reals $\{\varepsilon_n\}_{n=1}^{\infty}$ there is an increasing sequence of integers $\{n_k\}_{k=0}^{\infty}$ and a sequence

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of finite families of intervals $\{\mathcal{I}_n\}_{n=1}^\infty$ such that $|\mathcal{I}_n| \leq n$, $|I| < \varepsilon_n$ for every $I \in \mathcal{I}_n$ and $A \subseteq \bigcup_m \bigcap_{k \geq m} \bigcup \mathcal{I}_{n_k}$.

In [3] it was proved that every perfect measure zero set is permitted for any of the families \mathcal{N} , \mathcal{A} , \mathcal{N}_0 , and $p\mathcal{D}$; every γ -set has perfect measure zero; every perfect measure zero set has strong measure zero; the subgroup of \mathbb{R} , + generated by a set having perfect measure zero has perfect measure zero; and every set of cardinality less than the additivity of Lebesgue measure has perfect measure zero.

A result of [1] says that every set of cardinality less than \mathfrak{t} is permitted for \mathcal{N} . The cardinals \mathfrak{t} and the additivity of Lebesgue measure are two the best known lower bounds for the minimal size of a set not being permitted for \mathcal{N} . It is worth mentioning that these two bounds are mutually independent. (See a discussion in [3].) Some more bounds of cardinal invariants of other families of trigonometric thin sets can be found in [2] and [1]. Using some ideas of [1] we prove the following result.

Main Theorem 1.1 *Let \mathcal{F} be any of the families \mathcal{N} , \mathcal{A} , \mathcal{N}_0 and $p\mathcal{D}$. The unions of less than \mathfrak{t} sets having perfect measure zero are permitted for \mathcal{F} .*

Let us recall that \mathfrak{t} is the minimal size of a tower of subsets of ω , \mathfrak{h} is the distributivity number of the Boolean algebra $\mathcal{P}(\omega)/fin$, and \mathfrak{b} is the minimal size of an unbounded family of functions under the eventual dominance. It is well known that $\mathfrak{t} \leq \mathfrak{h} \leq \mathfrak{b}$.

2 The Proof of the Case \mathcal{N}

Let E be an N -set, i.e. there is a sequence $\{\rho_n\}_{n=1}^\infty$ such that $\sum_{n=1}^\infty \rho_n = \infty$ and $E = \{x \in \mathbb{R} : \sum_{n=1}^\infty \rho_n |\sin n\pi x| < \infty\}$. Let us denote $s_n = \sum_{k=1}^n \rho_k$. There is a surjective monotone function $\iota : \omega \rightarrow \omega \setminus \{0\}$ such that

$$\sum_{n=1}^{\infty} \frac{\rho_n}{s_n^{1+1/\iota^2(n)}} < \infty.$$

Let $\rho'_n = \rho_n/s_n$ and $\delta_n = s_n^{-1/\iota^2(n)}$. Then $\sum_{n=1}^\infty \rho'_n = \infty$ and $\sum_{n=1}^\infty \rho'_n \delta_n < \infty$. There is an increasing function $f \in {}^\omega\omega$ such that $\sum_{n=f(k)}^{f(k+1)-1} \rho'_n \geq 1$ for all $k \in \omega$. Let us denote $J_k = [f(k), f(k+1))$ and

$$\varepsilon_m = \min \left\{ \frac{\delta_n}{ns_n} : (\exists k) \iota(k) = m \ \& \ n \in J_k \right\}. \quad (2.1)$$

Lemma 2.1 *Suppose $Z \subseteq \omega$ is a finite set, $0 < \delta \leq 1$ and $x_1, \dots, x_n \in \mathbb{R}$. There is $Z' \subseteq Z$ such that $|Z'| \geq \delta^n |Z|$ and*

$$(\forall i, j \in Z') (\forall l = 1, \dots, n) |\sin(i - j)\pi x_l| < 2\pi\delta.$$

PROOF. First let $n = 1$. Find m such that $1/(2m) < \delta \leq 1/m$ and divide the interval $[0, 1]$ into m subintervals of length $1/m$. There is a set $Z' \subseteq Z$ such that $|Z'| \geq |Z|/m \geq \delta|Z|$ and for all $i \in Z'$, $\{ix_1\}$ are in the same subinterval. (Let us recall that $\{y\}$ denotes the fractional part of a real y .) Hence $|\sin(i - j)\pi x_1| \leq \pi/m < 2\pi\delta$ for all $i, j \in Z'$. Now in case $n > 1$ apply the previous result n times. \square

Lemma 2.2 *There is a system of functions $f_\alpha \in {}^\omega\mathbb{R}$ for $\alpha < \mathfrak{b}$ such that*

- (1) $(\forall^\infty n) 0 \leq f_\alpha(n) \leq s_n \delta_n^{\iota(n)}$.
- (2) $(\forall k \in \omega) \lim_{n \rightarrow \infty} f_\alpha(n) \delta_n^{k\iota(n)} = \infty$.
- (3) *If $\beta < \alpha < \mathfrak{b}$, then $(\forall^\infty n) f_\alpha(n) \leq f_\beta(n) \delta_n^{\iota(n)}$.*

PROOF. Let us set $f_0(n) = s_n \delta_n^{\iota(n)}$ and similarly in the non-limit steps let us set $f_{\alpha+1}(n) = f_\alpha(n) \delta_n^{\iota(n)}$. Then we get

$$\lim_{n \rightarrow \infty} f_0(n) \delta_n^{k\iota(n)} = \lim_{n \rightarrow \infty} s_n^{1-(k+1)/\iota(n)} \geq \lim_{n \rightarrow \infty} s_n^{1/2} = \infty$$

and condition (2) can be easily verified also for the non-limit step. Let $\alpha < \mathfrak{b}$ be limit. For each $\beta < \alpha$ there is $g_\beta \in {}^\omega\omega$ with $\lim_{n \rightarrow \infty} g_\beta(n) = \infty$ such that $\lim_{n \rightarrow \infty} f_\beta(n) \delta_n^{g_\beta(n)\iota(n)} = \infty$ (by (2) in the induction hypothesis). Let $g \in {}^\omega\omega$ be such that $\lim_{n \rightarrow \infty} g(n) = \infty$ and $(\forall \beta < \alpha) (\forall^\infty n) g(n) \leq g_\beta(n)$. Then for any $\beta < \alpha$, $\lim_{n \rightarrow \infty} f_\beta(n) \delta_n^{g(n)\iota(n)} = \infty$. We can find $h \in {}^\omega\omega$ with $\lim_{n \rightarrow \infty} h(n) = \infty$ such that $(\forall \beta < \alpha) (\forall^\infty n) h(n) \leq f_\beta(n) \delta_n^{g(n)\iota(n)}$. Let us set $f_\alpha(n) = h(n) \delta_n^{-g(n)\iota(n)}$. Then

$$\lim_{n \rightarrow \infty} f_\alpha(n) \delta_n^{k\iota(n)} = \lim_{n \rightarrow \infty} h(n) \delta_n^{(k-g(n))\iota(n)} \geq \lim_{n \rightarrow \infty} h(n) = \infty$$

for all $k \in \omega$ and

$$(\forall^\infty n) f_\alpha(n) = h(n) \delta_n^{-g(n)\iota(n)} \leq f_{\beta+1}(n) \leq f_\beta(n) \delta_n^{\iota(n)}$$

for all $\beta < \alpha$. \square

Lemma 2.3 *Let $\kappa < \mathfrak{h}$, let A_α , $\alpha < \kappa$, be perfectly measure zero sets, let a be an infinite subset of ω and let $\{\varepsilon_n\}_{n=1}^\infty$ be a given sequence of positive reals. There is a sequence of integers $\{n_k\}_{k=0}^\infty$ elements of a and a system of*

sequences of finite systems of intervals $\{\mathcal{I}_n^\alpha\}_{n=1}^\infty$, $\alpha < \kappa$, such that $|\mathcal{I}_n^\alpha| \leq n$, $(\forall I \in \mathcal{I}_n^\alpha) |I| < \varepsilon_n$ and $A_\alpha \subseteq \bigcup_m \bigcap_{k \geq m} \bigcup \mathcal{I}_{n_k}^\alpha$ for each α . (The sequence $\{n_k\}_{k=0}^\infty$ is independent of α .)

PROOF. Let \mathfrak{J} be the family of all infinite sequences of finite families of intervals $\{\mathcal{I}_n\}_{n=1}^\infty$ such that $|\mathcal{I}_n| \leq n$ and $(\forall I \in \mathcal{I}_n) |I| < \varepsilon_n$. The sets

$$D_\alpha = \left\{ b \in [\omega]^\omega : (\exists \{\mathcal{I}_n\}_{n=1}^\infty \in \mathfrak{J}) A_\alpha \subseteq \bigcup_m \bigcap_{n \in b \setminus m} \bigcup \mathcal{I}_n \right\}$$

are open dense in $[\omega]^\omega$. (See the proof of Theorem 1.2 (iii) of [3].) Choose $b \subseteq a$, $b \in \bigcap_{\alpha < \kappa} D_\alpha$ and let $\{n_k\}_{k=0}^\infty$ be the increasing enumeration of the set b . □

The last lemma enables us to choose n_k uniformly for a given family of perfect measure zero sets and so for the case of N -sets in the Main Theorem it is enough to prove the following proposition. Notice that in the proposition $\iota(n_k)$ can be replaced by n_k . (But it is easier handling terms $\iota(n_k)$ in the proof.)

Proposition 2.4 *Let $\{n_k\}_{k=0}^\infty$ be a given increasing sequence of integers and $\{\varepsilon_n\}_{n=1}^\infty$ be the sequence of positive reals defined by (2.1) for a given N -set E . Whenever $\nu < \mathfrak{t}$ and $\{\mathcal{I}_n^\alpha\}_{n=1}^\infty$, $\alpha \leq \nu$, are sequences of finite families of intervals such that $|\mathcal{I}_n| \leq n$ and $(\forall I \in \mathcal{I}_n) |I| \leq \varepsilon_n$, then $E \cup \bigcup_{\alpha \leq \nu} \bigcup_m \bigcap_{k \geq m} \bigcup \mathcal{I}_{i(n_k)}^\alpha$ is an N -set.*

PROOF. Let $\mathcal{I}_n^\alpha = \{[a_{n,l}^\alpha, a_{n,l}^\alpha + \varepsilon_n]\}_{l=1}^n$ for $n \geq 1$ and $\alpha \leq \nu$ and let us set $P_\alpha = \bigcup_m \bigcap_{k \geq m} \bigcup \mathcal{I}_{i(n_k)}^\alpha$. By induction we build $\{\varphi_\alpha : \alpha \leq \nu\}$ and $\{a_\alpha : \alpha \leq \nu\} \subseteq [\omega]^\omega$ such that

- (1) $\varphi_\alpha : \bigcup_{k \in a_\alpha} J_{n_k} \rightarrow [\omega]^{<\omega}$.
- (2) $(\forall k \in a_\alpha) (\forall n \in J_{n_k}) (\forall i, j \in \varphi_\alpha(n)) (\forall l = 1, \dots, \iota(n_k))$
 $|\sin(i - j)n\pi a_{i(n_k),l}^\alpha| < 2\pi\delta_n$.
- (3) $(\forall n \in \text{dom}(\varphi_\alpha)) \max \varphi_\alpha(n) \leq s_n$.
- (4) $(\forall^\infty n \in \text{dom}(\varphi_\alpha)) |\varphi_\alpha(n)| \geq f_\alpha(n)$.
- (5) If $\beta \leq \alpha$, then $a_\alpha \subseteq^* a_\beta$ and $(\forall^\infty n \in \text{dom}(\varphi_\alpha)) \varphi_\alpha(n) \subseteq \varphi_\beta(n)$.

For $\alpha = 0$ set $a_0 = \omega$ and for $n \in J_{n_k}$. Using Lemma 2.1 find $\varphi_0(n) \subseteq \{i : i \leq s_n\}$ so that $|\varphi_0(n)| \geq s_n \delta_n^{\iota(n_k)} \geq s_n \delta_n^{\iota(n)} \geq f_0(n)$ and

$$(\forall i, j \in \varphi_0(n)) |\sin(i - j)n\pi a_{i(n_k),l}^0| < 2\pi\delta_n$$

(whenever $\delta_n \leq 1$). Similarly using Lemma 2.1 we can find $\varphi_{\alpha+1}(n) \subseteq \varphi_\alpha(n)$ so that (2) and (4) hold and we can set $a_{\alpha+1} = a_\alpha$. Let $\alpha \leq \nu$ be limit. For $k \in \omega$ let

$$U_k = \{F \in J_{n_k}([\omega]^{<\omega}) : (\forall n \in \text{dom}(F)) \max F(n) \leq s_n \\ \& |F(n)| \geq f_\alpha(n) \& (\forall i, j \in F(n)) (\forall l = 1, \dots, \iota(n_k)) \\ |\sin(i - j)\pi n a_{\iota(n_k), l}^\alpha| < 2\pi\delta_n\}.$$

The set $U = \bigcup_{k \in \omega} U_k$ is countably infinite. For $\beta < \alpha$ let

$$X_\beta = \{F \in U : (\forall n \in \text{dom}(F)) F(n) \subseteq \varphi_\beta(n)\}.$$

For all $n \in J_{n_k}$, $\iota(n) \geq \iota(n_k)$. Hence for all but finitely many k for all $n \in J_{n_k}$, $f_\alpha(n) \leq f_\beta(n)\delta_n^{\iota(n)} \leq f_\beta(n)\delta_n^{\iota(n_k)}$ and using Lemma 2.1 we easily see that X_β is infinite. Moreover, whenever $\beta < \gamma < \alpha$, then $X_\gamma \subseteq^* X_\beta$. Since $\alpha < \mathfrak{t}$, there is $X \subseteq U$ such that $X \subseteq^* X_\beta$ for all $\beta < \alpha$. Since each U_k is finite we can choose X so that $|X \cap U_k| \leq 1$ for each $k \in \omega$. Let us define $\varphi_\alpha = \bigcup X$ and $a_\alpha = \{k : X \cap U_k \neq \emptyset\}$.

Let $\varphi = \varphi_\nu$ and let $a \subseteq \omega$ be the set of all $k \in a_\nu$ such that for each $n \in J_{n_k}$ there are $i_n, j_n \in \varphi(n)$, $j_n < i_n$, and for each such n let us put $\lambda_n = i_n - j_n$; $\lambda_n \leq s_n$. The set a contains all but finitely many $k \in a_\nu$. We prove that the series

$$\sum_{k \in a} \sum_{n \in J_{n_k}} \rho'_n |\sin \lambda_n n \pi x|$$

converges for $x \in E \cup \bigcup_{\alpha \leq \nu} P_\alpha$. This finishes the proof since

$$\sum_{k \in a} \sum_{n \in J_{n_k}} \rho'_n = \infty.$$

First notice that by (2) for all but finitely many $k \in a$ and for all $n \in J_{n_k}$

$$|\sin \lambda_n n \pi a_{\iota(n_k), l}^\alpha| \leq 2\pi\delta_n \quad \text{for } l = 1, \dots, \iota(n_k)$$

and since

$$|\sin \lambda_n n \pi \varepsilon_{\iota(n_k)}| \leq \lambda_n n \varepsilon_{\iota(n_k)} \leq s_n n \varepsilon_{\iota(n_k)} \leq \delta_n$$

we obtain that

$$|\sin \lambda_n n \pi x| \leq 3\pi\delta_n \quad \text{for } x \in \bigcup \mathcal{I}_{\iota(n_k)}^\alpha \text{ and } n \in J_{n_k} \tag{2.2}$$

holds for all but finitely many $k \in a$.

Let $x \in P_\alpha$, $\alpha \leq \nu$. There is m such that $x \in \bigcup \mathcal{I}_{l(n_k)}^\alpha$ and (2.2) holds for all $k \in a \setminus m$. Then

$$\sum_{k \in a \setminus m} \sum_{n \in J_{n_k}} \rho'_n |\sin \lambda_n n \pi x| \leq \sum_{k \in a \setminus m} \sum_{n \in J_{n_k}} 3\rho'_n \delta_n < \infty.$$

If $x \in E$, then

$$\sum_{k \in a} \sum_{n \in J_{n_k}} \rho'_n |\sin \lambda_n n \pi x| \leq \sum_{k \in a} \sum_{n \in J_{n_k}} \rho'_n \lambda_n |\sin n \pi x| \leq \sum_{n=1}^\infty \rho_n |\sin n \pi x| < \infty.$$

□

3 The Proof of the Cases $p\mathcal{D}$, \mathcal{N}_0 , \mathcal{A}

All these proofs are the same and we will outline the case of N_0 -sets. Let $E = \{x : \sum_{k=0}^\infty |\sin m_k \pi x| < \infty\}$ with $\{m_k\}_{k=0}^\infty$ strictly increasing. Let $\delta_n = 1/n^2$ and $\varepsilon = \delta_n/m_k$ for $k = \sum_{j=1}^n (1/\delta_j)^{j^2}$. The particular steps of the proof are analogous to the case of N -sets.

Lemma 3.1 *There is a system of functions $f_\alpha \in {}^\omega \mathbb{R}$ for $\alpha < \mathfrak{b}$ such that*

- (1) $(\forall^\infty n) 0 \leq f_\alpha(n) \leq (1/\delta_n)^{n^2} \delta_n^n$.
- (2) $(\forall k \in \omega) \lim_{n \rightarrow \infty} f_\alpha(n) \delta_n^{kn} = \infty$.
- (3) *If $\beta < \alpha < \mathfrak{b}$, then $(\forall^\infty n) f_\alpha(n) \leq f_\beta(n) \delta_n^n$.*

PROOF. Same as the proof of Lemma 2.2. □

Again it is enough to prove the following proposition. (Notice that the same proposition holds for $p\mathcal{D}$ -sets and \mathcal{A} -sets. It is enough to consider either quasinormal or pointwise convergence in the definition of the set E instead of absolute convergence of a series.)

Proposition 3.2 *Let $\{n_k\}_{k=0}^\infty$ be a given increasing sequence of integers and $\nu < \mathfrak{t}$. If $\{\mathcal{I}_n^\alpha\}_{n=1}^\infty$, $\alpha \leq \nu$, are sequences of finite families of intervals such that $|\mathcal{I}_n| \leq n$ and $(\forall I \in \mathcal{I}_n) |I| \leq \varepsilon_n$, then $E \cup \bigcup_{\alpha \leq \nu} \bigcup_m \bigcap_{k \geq m} \bigcup \mathcal{I}_{n_k}^\alpha$ is an N_0 -set.*

PROOF. Let $\mathcal{I}_n^\alpha = \{[a_{n,l}^\alpha, a_{n,l}^\alpha + \varepsilon_n]\}_{l=1}^n$ and put

$$P_\alpha = \bigcup_m \bigcap_{k \geq m} \bigcup \mathcal{I}_{n_k}^\alpha.$$

In the same way as in the proof of Proposition 2.4 we build $\{\varphi_\alpha : \alpha \leq \nu\}$ and $\{a_\alpha : \alpha \leq \nu\} \subseteq [\omega]^\omega$ by induction so that

- (1) $\varphi_\alpha : \{n_k : k \in a_\alpha\} \rightarrow [\omega]^{<\omega}$.
- (2) $(\forall n \in \text{dom}(\varphi_\alpha))(\forall i, j \in \varphi_\alpha(n))(\forall l = 1, \dots, n) |\sin(m_i - m_j)\pi a_{n,l}^\alpha| < 2\pi\delta_n$.
- (3) $(\forall n \in \text{dom}(\varphi_\alpha))(\forall i \in \varphi_\alpha(n)) \sum_{j=1}^{n-1} (1/\delta_j)^{j^2} \leq i < \sum_{j=1}^n (1/\delta_j)^{j^2}$.
- (4) $(\forall^\infty n \in \text{dom}(\varphi_\alpha)) |\varphi_\alpha(n)| \geq f_\alpha(n)$.
- (5) If $\beta \leq \alpha$, then $a_\alpha \subseteq^* a_\beta$ and $(\forall^\infty n \in \text{dom}(\varphi_\alpha)) \varphi_\alpha(n) \subseteq \varphi_\beta(n)$.

For $\alpha \leq \nu$ limit let U_n be the set of all pairs (n, F) such that $F \in [\omega]^{<\omega}$, $|F| \geq f_\alpha(n)$, $(\forall i \in F) \sum_{j=1}^{n-1} (1/\delta_j)^{j^2} \leq j < \sum_{j=1}^n (1/\delta_j)^{j^2}$, and $(\forall i, j \in F)(\forall l = 1, \dots, n) |\sin(m_i - m_j)\pi a_{n,l}^\alpha| < 2\pi\delta_n$. The set $U = \bigcup_{n \in \omega} U_n$ is countably infinite and for $\beta < \alpha$ the sets $X_\beta = \{(n, F) : F \subseteq \varphi_\beta(n)\}$ are infinite and decreasing with respect to \subseteq^* . There is an infinite set $X \subseteq U$ such that $X \subseteq^* X_\beta$ for all $\beta < \alpha$ and $|X \cap U_n| \leq 1$ for each n . Let us set $\varphi_\alpha = X$ and $a_\alpha = \{k : U_{n_k} \cap X \neq \emptyset\}$.

Let $\varphi = \varphi_\nu$ and let a be the set of all $k \in a_\nu$ for which there are two different members $i_k, j_k \in \varphi(n_k)$, $j_k < i_k$, and for each such k let us set $\lambda_k = m_{i_k} - m_{j_k}$. Notice that both sequences $\{i_k\}_{k=0}^\infty$ and $\{j_k\}_{k=0}^\infty$ are strictly increasing. Given $\alpha \leq \nu$ for all but finitely many $k \in a$, $n_k \in \text{dom}(\varphi_\alpha)$. Hence for all but finitely many $k \in a$

$$|\sin \lambda_k \pi a_{n_k, l}^\alpha| \leq 2\pi\delta_{n_k} \quad \text{for } l = 1, \dots, n_k$$

and

$$|\sin \lambda_k \pi \varepsilon_{n_k}| \leq (m_{i_k} - m_{j_k})\pi \varepsilon_{n_k} \leq m_{i_k} \pi \varepsilon_{n_k} \leq \pi \delta_{n_k}.$$

Consequently

$$|\sin \lambda_k \pi x| \leq 3\pi\delta_{n_k} \quad \text{for } x \in \bigcup \mathcal{I}_{n_k}^\alpha \quad (3.1)$$

for all but finitely many $k \in a$. We prove that $\sum_{k \in a} |\sin \lambda_k \pi x|$ converges for $x \in E \cup \bigcup_{\alpha \leq \nu} P_\alpha$.

Let $x \in P_\alpha$. There is m such that $x \in \bigcup \mathcal{I}_{n_k}^\alpha$ for all $k \geq m$ and such that (3.1) holds for all $k \in a \setminus m$. Then $\sum_{k \in a \setminus m} |\sin \lambda_k \pi x| \leq \sum_{k \in a \setminus m} 3\pi\delta_{n_k} < \infty$. For $x \in E$, $\sum_{k \in a} |\sin \lambda_k \pi x| \leq \sum_{k \in a} |\sin m_{i_k} \pi x| + \sum_{k \in a} |\sin m_{j_k} \pi x| < \infty$. \square

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