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## ON TREE IDEALS

MARTIN GOLDSTERN, MIROSLAV REPICKÝ, SAHARON SHELAH,  
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(Communicated by Andreas R. Blass)

**ABSTRACT.** Let  $l^0$  and  $m^0$  be the ideals associated with Laver and Miller forcing, respectively. We show that  $\mathbf{add}(l^0) < \mathbf{cov}(l^0)$  and  $\mathbf{add}(m^0) < \mathbf{cov}(m^0)$  are consistent. We also show that both Laver and Miller forcing collapse the continuum to a cardinal  $\leq \mathfrak{h}$ .

### INTRODUCTION AND NOTATION

In this paper we investigate the ideals connected with the classical tree forcings introduced by Laver [La] and Miller [Mi]. Laver forcing  $\mathbb{L}$  is the set of all trees  $p$  on  ${}^{<\omega}\omega$  such that  $p$  has a stem and whenever  $s \in p$  extends  $\mathit{stem}(p)$  then  $\mathit{Succ}_p(s) := \{n : s \hat{\ } n \in p\}$  is infinite. Miller forcing  $\mathbb{M}$  is the set of all trees  $p$  on  ${}^{<\omega}\omega$  such that  $p$  has a stem and for every  $s \in p$  there is  $t \in p$  extending  $s$  such that  $\mathit{Succ}_p(t)$  is infinite. We denote the set of all these splitting nodes in  $p$  by  $\mathit{Split}(p)$ . For any  $t \in \mathit{Split}(p)$ ,  $\mathit{Split}_p(t)$  is the set of all minimal (with respect to extension) members of  $\mathit{Split}(p)$  which properly extend  $t$ . For both  $\mathbb{L}$  and  $\mathbb{M}$  the order is inclusion.

The Laver ideal  $l^0$  is the set of all  $X \subseteq {}^\omega\omega$  with the property that for every  $p \in \mathbb{L}$  there is  $q \in \mathbb{L}$  extending  $p$  such that  $X \cap [q] = \emptyset$ . Here  $[q]$  denotes the set of all branches of  $q$ . The Miller ideal  $m^0$  is defined analogously, using conditions in  $\mathbb{M}$  instead of  $\mathbb{L}$ . By a fusion argument one easily shows that  $l^0$  and  $m^0$  are  $\sigma$ -ideals.

The additivity (**add**) of any ideal is defined as the minimal cardinality of a family of sets belonging to the ideal whose union does not. The covering number (**cov**) is defined as the least cardinality of a family of sets from the ideal whose union is the whole set on which the ideal is defined— ${}^\omega\omega$  in our case. Clearly  $\omega_1 \leq \mathbf{add}(l^0) \leq \mathbf{cov}(l^0) \leq \mathfrak{c}$  and  $\omega_1 \leq \mathbf{add}(m^0) \leq \mathbf{cov}(m^0) \leq \mathfrak{c}$  hold.

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The main result in this paper says that there is a model of ZFC where  $\mathbf{add}(l^0) < \mathbf{cov}(l^0)$  and  $\mathbf{add}(m^0) < \mathbf{cov}(m^0)$  hold. The motivation was that by a result of Plewik [P1] it was known that the additivity and the covering number of the ideal connected with Mathias forcing are the same and they are equal to the cardinal invariant  $\mathfrak{h}$ —the least cardinality of a family of maximal antichains of  $\mathcal{P}(\omega)/fin$  without a common refinement. On the other hand, in [JuMiSh] it was shown that  $\mathbf{add}(s^0) < \mathbf{cov}(s^0)$  is consistent, where  $s^0$  is Marczewski’s ideal—the ideal connected with Sacks forcing  $\mathbb{S}$ . Intuitively,  $\mathbb{L}$  and  $\mathbb{M}$  sit somewhere between Mathias forcing and  $\mathbb{S}$ . In [GoJoSp] it was shown that under Martin’s axiom  $\mathbf{add}(l^0) = \mathbf{add}(m^0) = \mathfrak{c}$ , whereas this is false for  $s^0$  (see [JuMiSh]).

The method of proof for  $\mathbf{add}(s^0) < \mathbf{cov}(s^0)$  in [JuMiSh] is the following: For a forcing  $P$  denote by  $\kappa(P)$  the least cardinal to which forcing with  $P$  collapses the continuum. In [JuMiSh] it is shown that  $\mathbf{add}(s^0) \leq \kappa(\mathbb{S})$ . In [BaLa] it was shown that in  $V^{\mathbb{S}_{\omega_2}} \kappa(\mathbb{S}) = \omega_1$  holds, where  $\mathbb{S}_{\omega_2}$  is the countable support iteration of length  $\omega_2$  of  $\mathbb{S}$ . Hence  $V^{\mathbb{S}_{\omega_2}} \models \mathbf{add}(s^0) = \omega_1$ . On the other hand, a Löwenheim-Skolem argument shows that  $V^{\mathbb{S}_{\omega_2}} \models \mathbf{cov}(s^0) = \omega_2$ .

Our method of proof is similar. Denoting by  $P_{\omega_2}$  a countable support iteration of length  $\omega_2$  of  $\mathbb{L}$  and  $\mathbb{M}$  (each occurring on a stationary set), in §2 we prove the following:

**Theorem.**

$$V^{P_{\omega_2}} \models \omega_1 = \mathbf{add}(l^0) = \mathbf{add}(m^0) < \mathbf{cov}(l^0) = \mathbf{cov}(m^0) = \omega_2.$$

The crucial steps in the proof are to show that  $\kappa(\mathbb{L}), \kappa(\mathbb{M})$  equal  $\omega_1$  and  $\mathbf{add}(l^0) \leq \kappa(\mathbb{L}), \mathbf{add}(m^0) \leq \kappa(\mathbb{M})$  hold.

We will use the standard terminology for set theory and forcing. By  $\mathfrak{b}$  we denote the least cardinality of a family of functions in  ${}^\omega\omega$  which is unbounded with respect to eventual dominance and  $\mathfrak{d}$  will be the least cardinality of a dominating family in  ${}^\omega\omega$ . Moreover,  $\mathfrak{p}$  is the least cardinality of a filter base on  $([\omega]^\omega, \subseteq^*)$  without any lower bound, and  $\mathfrak{t}$  is the least cardinality of a decreasing chain in  $([\omega]^\omega, \subseteq^*)$  without any lower bound. It is easy to see that  $\omega_1 \leq \mathfrak{p} \leq \mathfrak{t} \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$ .

1. UPPER AND LOWER BOUNDS

**Theorem 1.1.** (1)  $\mathfrak{t} \leq \mathbf{add}(l^0) \leq \mathbf{cov}(l^0) \leq \mathfrak{b}$ .

(2)  $\mathfrak{p} \leq \mathbf{add}(m^0) \leq \mathbf{cov}(m^0) \leq \mathfrak{d}$ .

*Proof of Theorem 1.1(1).* We have to prove the first and the third inequality. For the third inequality, let  $\langle f_\alpha : \alpha < \mathfrak{b} \rangle$  be an unbounded family. Define

$$X_\alpha := \{f \in {}^\omega\omega : (\exists^\infty k) f(k) < f_\alpha(k)\}.$$

Clearly  $\bigcup \{X_\alpha : \alpha < \mathfrak{b}\} = {}^\omega\omega$ . We claim  $X_\alpha \in l^0$ . Let  $p \in \mathbb{L}$ . We define  $q \in \mathbb{L}$  as follows:  $stem(q) := stem(p)$ , and for any  $s$  extending  $stem(q)$  we have  $s \in q$  if and only if  $s \in p$  and  $(\forall k)$  if  $|stem(q)| \leq k < |s|$ , then  $s(k) \geq f_\alpha(k)$ . Then clearly  $q \in \mathbb{L}$ ,  $q$  extends  $p$ , and  $q \cap X_\alpha = \emptyset$ .

In order to prove the first inequality we use the following notation from [JuMiSh]: Let  $Q := \{\bar{A} = \langle A_s : s \in {}^{<\omega}\omega \rangle : (\forall s) A_s \in [\omega]^\omega\}$ . For  $\bar{A} \in Q$  we

define a sequence of Laver trees  $\langle p_s(\bar{A}) : s \in {}^{<\omega}\omega \rangle$  as follows:  $p_s(\bar{A})$  is the unique Laver tree such that  $stem(p_s(\bar{A})) = s$  and if  $t \in p_s(\bar{A})$  extends  $s$ , then  $Succ_{p_s(\bar{A})}(t) = A_t$ .

For  $\bar{A}, \bar{B} \in Q$  we define:

$$\begin{aligned} \bar{A} \subseteq \bar{B} &\Leftrightarrow (\forall s) A_s \subseteq B_s, \\ \bar{A} \subseteq^* \bar{B} &\Leftrightarrow (\forall s) A_s \subseteq^* B_s, \\ \bar{A} \leq^* \bar{B} &\Leftrightarrow (\forall s) A_s \subseteq^* B_s \wedge (\forall^\infty s) A_s \subseteq B_s. \end{aligned}$$

Here  $\leq^*$  is a slight but important modification of  $\subseteq^*$  from [JuMiSh].

**Fact 1.2.**  $(Q, \leq^*)$  is  $t$ -closed.

*Proof of Fact 1.2.* Suppose  $\langle \bar{A}_\alpha : \alpha < \gamma \rangle$ , where  $\gamma < t$  is a decreasing sequence in  $(Q, \leq^*)$ . Let  $\bar{A}_\alpha := \langle A_s^\alpha : s \in {}^{<\omega}\omega \rangle$ . Since  $\gamma < t$ , there is  $\bar{B}' = \langle B'_s : s \in {}^{<\omega}\omega \rangle \in Q$  such that  $(\forall \alpha < \gamma) \bar{B}' \subseteq^* \bar{A}_\alpha$ . Define  $f_\alpha : {}^{<\omega}\omega \rightarrow \omega$  such that  $(\forall s) B'_s \setminus f_s(\alpha) \subseteq A_s^\alpha$ . Since  $t \leq b$ , there exists  $f : {}^{<\omega}\omega \rightarrow \omega$  such that  $(\forall \alpha) (\forall^\infty s) f_\alpha(s) \leq f(s)$ . Now let  $B_s := B'_s \setminus f(s)$  and  $\bar{B} := \langle B_s : s \in {}^{<\omega}\omega \rangle$ . It is easy to check that  $(\forall \alpha < \gamma) \bar{B} \leq^* \bar{A}_\alpha$ .

**Fact 1.3.** Suppose  $X \in I^0$  and  $\bar{A} \in Q$ . There exists  $\bar{B} \in Q$  such that  $\bar{B} \subseteq \bar{A}$  and  $(\forall s \in {}^{<\omega}\omega) [p_s(\bar{B})] \cap X = \emptyset$ .

*Proof of Fact 1.3.* First note that if  $D := \{p \in \mathbb{L} : [p] \cap X = \emptyset\}$ , then  $D$  is open dense and even 0-dense, i.e., for every  $p \in \mathbb{L}$  there exists  $q \in D$  extending  $p$  such that  $stem(q) = stem(p)$ . The proof of this is similar to Laver's proof in [La] that the set of Laver trees deciding a sentence in the language of forcing with  $\mathbb{L}$  is 0-dense: Suppose  $p \in \mathbb{L}$  has no 0-extension whose branches are not in  $X$ . Then inductively we can construct  $q \in \mathbb{L}$  extending  $p$  such that every extension of  $q$  has a branch in  $X$ , contradicting  $X \in I^0$ .

Using this it is straightforward to construct  $\bar{B}$  as desired.

**Fact 1.4.** Suppose  $X \subseteq {}^\omega\omega$ ,  $\bar{A}, \bar{B} \in Q$ ,  $\bar{B} \leq^* \bar{A}$ , and  $(\forall s) [p_s(\bar{A})] \cap X = \emptyset$ . Then  $(\forall s) [p_s(\bar{B})] \cap X = \emptyset$ .

*Proof of Fact 1.4.* Clearly, if  $F \subseteq p_s(\bar{B})$  is finite, then

$$[p_s(\bar{B})] = \bigcup \{ [p_t(\bar{B})] : t \in p_s(\bar{B}) \setminus F \}.$$

But for almost all  $t \in p_s(\bar{B})$ ,  $p_t(\bar{B})$  extends  $p_t(\bar{A})$ . So clearly  $[p_s(\bar{B})] \subseteq [p_s(\bar{A})]$  and hence  $[p_s(\bar{B})] \cap X = \emptyset$ .

*End of the proof of Theorem 1.1(1).* Suppose we are given  $\langle X_\alpha : \alpha < \gamma \rangle$  and  $q \in \mathbb{L}$ , where  $\gamma < t$  and  $(\forall \alpha) X_\alpha \in I^0$ . Choose  $\bar{A} \in Q$  such that  $p_{stem(q)}(\bar{A}) = q$ , and let  $\bar{B}_0$  be the  $\bar{B}$  given by Fact 1.3 for  $\bar{A}$  and  $X_0$ . If  $\langle \bar{B}_\alpha : \alpha < \beta \rangle$  is constructed for  $\beta \leq \gamma$  and  $\beta$  is a successor, then choose  $\bar{B}_\beta$  as given by Fact 1.3 for  $\bar{A} = \bar{B}_{\beta-1}$  and  $X = X_\beta$ . If  $\beta$  is a limit, then by Fact 1.2 choose first  $\bar{A}$  such that  $(\forall \alpha < \beta) \bar{A} \leq^* \bar{B}_\alpha$  and then find  $\bar{B}_\beta \subseteq \bar{A}$  as given by Fact 1.3 for  $\bar{A}$  and  $X = X_\beta$ . Finally, if we have constructed  $\bar{B}_\gamma = \langle B'_s : s \in {}^{<\omega}\omega \rangle$ , define  $\bar{B} := \langle B_s : s \in {}^{<\omega}\omega \rangle$  by  $B_s := B'_s \cap Succ_q(s)$  if  $s \in q$  extends  $stem(q)$ , and  $B_s := B'_s$  otherwise. It is easy to check that  $\bar{B} \in Q$ ,  $p_{stem(q)}(\bar{B})$  extends  $q$  and  $(\forall \alpha < \gamma) [p_{stem(q)}(\bar{B})] \cap X_\alpha = \emptyset$ .

*Proof of Theorem 1.1(2).* The proof is similar to (1). For the third inequality, let  $\langle f_\alpha : \alpha < \mathfrak{d} \rangle$  be a dominating family. Define

$$X_\alpha := \{f \in {}^\omega\omega : (\forall^\infty k) f(k) < f_\alpha(k)\}.$$

Then  $\bigcup\{X_\alpha : \alpha < \mathfrak{d}\} = {}^\omega\omega$  and in an analogous way as in (1) it can be seen that  $X_\alpha \in m^0$ .

In order to prove the first inequality we need the following concept from [GoJoSp]. Let  $R$  be the set of all  $\bar{P} = \langle P_s : s \in {}^{<\omega}\omega \rangle$  where each  $P_s \subseteq {}^{<\omega}\omega$  is infinite,  $t \in P_s$  implies  $s \subset t$ , and if  $t, t' \in P_s$  are distinct, then  $t(|s|) \neq t'(|s|)$ . Given  $\bar{P} \in R$  we can define  $\langle p_s(\bar{P}) : s \in {}^{<\omega}\omega \rangle$  as follows:  $p_s(\bar{P})$  is the unique Miller tree with stem  $s$  such that if  $t \in \text{Split}_{p_s(\bar{P})}(t) = P_t$ .

Define the following relations on  $R$ :

$$\begin{aligned} \bar{P} \leq \bar{Q} &\Leftrightarrow (\forall s) p_s(\bar{P}) \leq p_s(\bar{Q}), \\ \bar{P} \approx \bar{Q} &\Leftrightarrow (\forall s) P_s =^* Q_s \wedge (\forall^\infty s) P_s = Q_s, \\ \bar{P} \leq^* \bar{Q} &\Leftrightarrow (\exists \bar{P}') \bar{P} \approx \bar{P}' \wedge \bar{P}' \leq \bar{Q}. \end{aligned}$$

**Fact 1.5** [GoJoSp, 4.14]. *Assume  $MA_\kappa(\sigma\text{-centered})$ . If  $\langle \bar{P}_\alpha : \alpha < \kappa \rangle$  is a  $\leq^*$ -decreasing sequence in  $R$ , then there exists  $\bar{Q} \in R$  such that  $(\forall \alpha < \kappa) \bar{Q} \leq^* \bar{P}_\alpha$ .*

The following two facts have proofs similar to those of Facts 1.3 and 1.4.

**Fact 1.6.** *Suppose  $X \in m^0$  and  $\bar{P} \in R$ . There exists  $\bar{Q} \leq \bar{P}$  such that  $(\forall s)[p_s(\bar{Q})] \cap X = \emptyset$ .*

**Fact 1.7.** *Suppose  $X \in m^0, \bar{P}, \bar{Q} \in R, \bar{P} \leq^* \bar{Q}$ , and  $(\forall s)[p_s(\bar{Q})] \cap X = \emptyset$ . Then  $(\forall s)[p_s(\bar{P})] \cap X = \emptyset$ .*

Now using, Facts 1.5, 1.6, 1.7 and the well-known result that for all  $\kappa < \mathfrak{p}$   $MA_\kappa(\sigma\text{-centered})$  holds, a similar construction as in Theorem 1.1(1) shows that  $\mathfrak{p} \leq \mathbf{add}(m^0)$ .

## 2. ADD AND COV ARE DISTINCT

**Definition 2.1.** A set  $A \subseteq {}^\omega\omega$  is called *strongly dominating* if and only if

$$(\forall f \in {}^\omega\omega)(\exists \eta \in A)(\forall^\infty k) f(\eta(k-1)) < \eta(k).$$

**Definition 2.2.** For any set  $A \subseteq {}^\omega\omega$ , we define the domination game  $D(A)$  as follows:

There are two players, GOOD and BAD. GOOD plays first. The game lasts  $\omega$  moves.

GOOD	BAD
$s$	$n_0$
$m_0$	$n_1$
$m_1$	$\vdots$
$\vdots$	$\vdots$

The rules are:  $s$  is a sequence in  ${}^{<\omega}\omega$ , and the  $n_i$  and  $m_i$  are natural numbers. (Whoever breaks these rules first, loses immediately.)

The GOOD player wins if and only if:

- (a) For all  $i$ ,  $m_i > n_i$ .
- (b) The sequence  $s \frown m_0 \frown m_1 \frown \dots$  is in  $A$ .

**Lemma 2.3.** *Let  $A \subseteq {}^\omega\omega$  be a Borel set. Then the following are equivalent:*

- (1) *There exists a Laver tree  $p$  such that  $[p] \subseteq A$ .*
- (2)  *$A$  is strongly dominating.*
- (3) *GOOD has a winning strategy in the game  $D(A)$ .*

*Remark.* Strongly dominating is not the same as dominating. For example, the closed set

$$A := \{\eta \in {}^\omega\omega : (\forall k)\eta(2k) = \eta(2k + 1)\}$$

is dominating but is not strongly dominating.

*Proof of Lemma 2.3.* We consider the following condition:

(4) (For all  $F : {}^{<\omega}\omega \times \omega \rightarrow \omega$ )  $(\exists \eta \in A)(\forall^\infty k)(\forall i \leq k)\eta(k) > F(\eta \upharpoonright k, i)$ . We will show (1)  $\rightarrow$  (2)  $\rightarrow$  (4)  $\rightarrow$  (3)  $\rightarrow$  (1).

- (1)  $\rightarrow$  (2) is clear.
- (2)  $\rightarrow$  (4): Given  $F$ , define  $f$  by

$$f(m) := \max\{F(s, i) : i \leq m, s \in m^{\leq m+1}\} + m;$$

$f$  is increasing,  $f(m) \geq m$  for all  $m$ .

Find  $\eta$  such that  $(\forall^\infty k)\eta(k) > f(\eta(k-1))$ . Then  $\eta$  is increasing. For almost all  $k$  we have, letting  $m := \eta(k-1)$ :  $m \geq k-1$ , so  $\eta \upharpoonright k \in m^{\leq m+1}$ , so by the definition of  $f$  we get  $f(m) \geq F(\eta \upharpoonright k, i)$  for any  $i \leq k$ . So  $\eta(k) > f(\eta(k-1)) \geq F(\eta \upharpoonright k, i)$ .

(4)  $\rightarrow$  (3): Assume that GOOD has no winning strategy. Then BAD has a winning strategy  $\sigma$  (since the game  $D(A)$  is Borel, hence determined).

We can find a function  $F : {}^{<\omega}\omega \times \omega \rightarrow \omega$  such that for all  $s, m_0, \dots, m_k$  we have

$$\sigma(s, m_0, \dots, m_k) = F(s \frown m_0 \frown \dots \frown m_k, |s|).$$

Find  $\eta \in A$  as in (4). So there is  $k_0$  such that  $\forall k \geq k_0 \eta(k) \geq F(\eta \upharpoonright k, k_0)$ . So in the play

GOOD	BAD
$s := \eta \upharpoonright k_0$	$n_0 := \sigma(s) = F(\eta \upharpoonright k_0, k_0)$
$m_0 := \eta(k_0 + 1)$	$n_1 := \sigma(s, m_0) = F(\eta \upharpoonright (k_0 + 1), k_0)$
$m_1 := \eta(k_0 + 2)$	$\vdots$
$\vdots$	$\vdots$

player BAD followed the strategy  $\sigma$ , but player GOOD won, a contradiction.

(3)  $\rightarrow$  (1): Let  $B$  be the set of all sequences  $s \frown m_0 \frown m_1 \frown \dots$  that can be played when GOOD follows a specific winning strategy. Clearly  $B \subseteq A$ , and for some Laver tree  $p$ ,  $B = [p]$ .

**Lemma 2.4** [Ke]. *Let  $A \subseteq {}^\omega\omega$  be an analytic set. Then the following are equivalent:*

- (1) *There exists a Miller tree  $p$  such that  $[p] \subseteq A$ .*
- (2)  *$A$  is unbounded in  $({}^\omega\omega, \leq^*)$ .*

**Lemma 2.5.** (1) *Suppose  $\mathfrak{b} = \mathfrak{c}$ . For every dense open  $D \subseteq \mathbb{L}$  there exists a maximal antichain  $A \subseteq D$  such that*

$$(*) \quad \forall q \in \mathbb{L} ([q] \subseteq \bigcup \{ [p] : p \in A \} \Rightarrow \exists A' \in [A]^{<\mathfrak{c}} \forall p \in A \setminus A' p \perp q).$$

- (2) *The same is true for  $\mathbb{M}$ .*

*Proof.* Let  $\mathbb{L} = \{q_\alpha : \alpha < \mathfrak{c}\}$ . Inductively we will define a set  $S \subseteq \mathfrak{c}$  and sequences  $\langle x_\gamma : \gamma < \mathfrak{c} \rangle$  and  $\langle p_\gamma : \gamma \in S \rangle$ . Finally we will let  $A = \{p_\gamma : \gamma \in S\}$ .

Let  $0 \in S$  and choose  $x_0 \in [q_0]$  arbitrarily.

It can easily be seen that every Laver tree contains  $\mathfrak{c}$  extensions such that every two of them do not contain a common branch. So clearly we may find  $p_0 \in D$  such that  $x_0 \notin [p_0]$ .

Now suppose that  $\langle x_\gamma : \gamma < \alpha \rangle$  and  $\langle p_\gamma : \gamma \in S \cap \alpha \rangle$  have been constructed for  $\alpha < \mathfrak{c}$ .

First choose  $x_\alpha \in [q_\alpha]$  arbitrarily, but such that, if  $[q_\alpha] \not\subseteq \bigcup \{ [p_\gamma] : \gamma < \alpha \}$ , then  $x_\alpha \notin \bigcup \{ [p_\gamma] : \gamma < \alpha \}$ .

In order to decide whether  $\alpha \in S$  or not we distinguish the following two cases:

*Case 1.*  $q_\alpha$  is compatible with some  $p_\gamma, \gamma < \alpha$ . In this case  $\alpha \notin S$ .

*Case 2.*  $q_\alpha$  is incompatible with all  $p_\gamma, \gamma < \alpha$ . Now we let  $\alpha \in S$ , and we define  $p_\alpha$  as follows:

By Lemma 2.3 for each  $\gamma \in \alpha$  we may find  $f_\gamma : \omega \rightarrow \omega$  such that

$$(**) \quad (\forall \eta \in [p_\gamma] \cap [q_\alpha]) (\exists^\infty k) \eta(k) \leq f_\gamma(\eta(k-1)).$$

By our assumption on  $\mathfrak{b}$  there exists a strictly increasing  $f$  which dominates all the  $f_\gamma$ 's. Now define  $p'_\alpha \in \mathbb{L}$  as follows:  $stem(p'_\alpha) = stem(q_\alpha)$ , and for  $t \in p'_\alpha$ , if  $t \supseteq stem(p'_\alpha)$  and  $|t| = n$ , we require

$$Succ_{p'_\alpha}(t) = Succ_{q_\alpha}(t) \cap [f(t(n-1)), \infty).$$

Clearly  $p'_\alpha \in \mathbb{L}$ ,  $p'_\alpha \subseteq q_\alpha$ , and by  $(**)$  and our assumption on  $f$  we conclude  $[p_\gamma] \cap [p'_\alpha] = \emptyset$  for every  $\gamma < \alpha$ .

By the remark above that every Laver tree contains  $\mathfrak{c}$  extensions such that every two of them do not contain a common branch, we may find  $p_\alpha \in D$  such that  $p_\alpha$  extends  $p'_\alpha$  and  $[p_\alpha]$  and  $\{x_\gamma : \gamma \leq \alpha\}$  are disjoint.

This finishes the construction. Now let  $A := \{p_\gamma : \gamma \in S\}$ .

Since every  $q_\alpha$  is either compatible with some  $p_\gamma, \gamma < \alpha$  (Case 1) or contains the condition  $p_\alpha$  (Case 2), and for  $\alpha \neq \gamma$  with  $\alpha, \gamma \in S$  we have  $[p_\alpha] \cap [p_\gamma] = \emptyset$ , we conclude that  $A$  is a maximal antichain.

$A$  also satisfies condition  $(*)$ : Let  $q = q_\alpha$ . By construction, if  $[q_\alpha] \not\subseteq \bigcup \{ [p_\gamma] : \gamma \in S \cap \alpha \}$ , then  $[q_\alpha] \not\subseteq \bigcup \{ [p_\gamma] : \gamma \in S \}$ .

The proof of (2) is analogous, but instead of Lemma 2.3 we use Lemma 2.4.

**Lemma 2.6.** *Suppose  $\mathfrak{b} = \mathfrak{c}$ . Then  $\mathbf{add}(l^0) \leq \kappa(\mathbb{L})$  and  $\mathbf{add}(m^0) \leq \kappa(\mathbb{M})$ .*

*Proof.* We may assume  $\kappa(\mathbb{L}) < \mathfrak{c}$ . Let  $\dot{f}$  be a  $\mathbb{L}$ -name such that  $\Vdash_{\mathbb{L}} \text{“}\dot{f} : \kappa(\mathbb{L}) \rightarrow \mathfrak{c} \text{ is onto”}$ . For  $\alpha < \kappa(\mathbb{L})$  let

$$D_\alpha := \{p \in \mathbb{L} : (\exists \beta)p \Vdash_{\mathbb{L}} \dot{f}(\alpha) = \beta\}.$$

For  $p \in D_\alpha$  we write  $\beta_p = \beta_p(\alpha)$  for the unique  $\beta$  satisfying  $p \Vdash_{\mathbb{L}} \dot{f}(\alpha) = \beta$ .

Clearly  $D_\alpha$  is dense and open. So we may choose a maximal antichain  $A_\alpha \subseteq D_\alpha$  as in Lemma 2.5. Let

$$X_\alpha := {}^\omega\omega \setminus \bigcup \{[p] : p \in A_\alpha\}.$$

Then  $X_\alpha \in I^0$ . We claim that  $X = \bigcup_{\alpha < \kappa(\mathbb{L})} X_\alpha \notin I^0$ . Suppose on the contrary  $X \in I^0$ . So we may find  $q \in \mathbb{L}$  such that  $[q] \cap X = \emptyset$  and hence  $[q] \subseteq \bigcup \{[p] : p \in A_\alpha\}$  for each  $\alpha$ . By the choice of  $A_\alpha$  each of the sets

$$B_\alpha := \{\beta_p(\alpha) : p \in A_\alpha, p \text{ compatible with } q\}$$

is bounded in  $\mathfrak{c}$ . Since  $\mathfrak{c}$  is regular by our assumption  $\mathfrak{b} = \mathfrak{c}$ , we can find  $\nu < \mathfrak{c}$  such that for all  $\alpha < \kappa(\mathbb{L})$ ,  $B_\alpha \subseteq \nu$ . So easily conclude that

$$q \Vdash_{\mathbb{L}} \text{“}\text{ran}(\dot{f}) \subseteq \nu < \mathfrak{c}\text{”}.$$

This is a contradiction.

The proof for  $\mathbb{M}$  is similar.

**Theorem 2.7.**  $\kappa(\mathbb{L}) \leq \mathfrak{h}$  and  $\kappa(\mathbb{M}) \leq \mathfrak{h}$ .

*Proof.* We prove it only for  $\mathbb{L}$ . The proof for  $\mathbb{M}$  is very similar. We work in  $V$ . Let  $\langle \mathcal{A}_\alpha : \alpha < \mathfrak{h} \rangle$  be a family of maximal almost disjoint families such that:

- (1) if  $\alpha < \beta < \mathfrak{c}$ , then  $\mathcal{A}_\beta$  refines  $\mathcal{A}_\alpha$ ;
- (2) there exists no maximal almost disjoint family refining all the  $\mathcal{A}_\alpha$ ;
- (3)  $\bigcup \{\mathcal{A}_\alpha : \alpha < \mathfrak{h}\}$  is dense in  $([\omega]^\omega, \subseteq^*)$ .

That such a sequence exists was shown in [BaPeSi].

Since  $\mathfrak{h}$  is regular, for every  $p \in \mathbb{L}$  there exists  $\alpha < \mathfrak{h}$  such that for each  $s \in \text{Split}(p)$  there is  $A \in \mathcal{A}_\alpha$  with  $A \subseteq^* \text{Succ}_p(s)$ . Hence, writing  $\mathbb{L}_\alpha$  for the set of those  $p \in \mathbb{L}$  for which  $\alpha$  has the property just stated, we conclude  $\mathbb{L} = \bigcup \{\mathbb{L}_\alpha : \alpha < \mathfrak{h}\}$ .

For each  $A \in \mathcal{A}_\alpha$  choose  $\mathcal{B}_A = \{B^A(p) : p \in \mathbb{L}\}$ , a maximal almost disjoint family on  $A$ .

Now we will define  $\mathbb{L}'_\alpha := \{q^\alpha(p) : p \in \mathbb{L}_\alpha\}$  such that  $q^\alpha(p)$  extends  $p$  for every  $p \in \mathbb{L}_\alpha$  and  $p_1 \neq p_2$  implies  $q^\alpha(p_1) \perp q^\alpha(p_2)$ . For  $p \in \mathbb{L}_\alpha$ ,  $q^\alpha(p)$  will be defined as follows:

For each  $s \in \text{Split}(p)$  let  $C_s^\alpha(p) := \text{Succ}_p(s) \cap B^A(p)$  where  $A \in \mathcal{A}_\alpha$  is such that  $A \subseteq^* \text{Succ}_p(s)$ . So clearly  $C_s^\alpha(p)$  is infinite. Now  $q^\alpha(p)$  is the unique Laver tree  $\leq p$  satisfying  $\text{stem}(q^\alpha(p)) = \text{stem}(p)$  and for each  $s \in \text{Split}(q^\alpha(p))$  we have  $\text{Succ}_{q^\alpha(p)}(s) = C_s^\alpha(p)$ .

It is not difficult to see that  $\mathbb{L}'_\alpha$  has the stated properties.

Now we are ready to define a  $\mathbb{L}$ -name  $\dot{f}$  such that  $\Vdash_{\mathbb{L}} \text{“}\dot{f} : \mathfrak{h}^V \rightarrow \mathfrak{c}^V \text{ is onto”}$ : For each  $p \in \mathbb{L}_\alpha$ , let  $\{r_\xi^\alpha(p) : \xi < \mathfrak{c}\} \subseteq \mathbb{L}$  be a maximal antichain below  $q^\alpha(p)$ , and define  $\dot{f}$  in such a way that  $r_\xi^\alpha(p) \Vdash_{\mathbb{L}} \text{“}\dot{f}(\alpha) = \xi\text{”}$ . As  $\bigcup \{\mathbb{L}'_\alpha : \alpha < \mathfrak{h}\}$  is dense in  $\mathbb{L}$ , it is easy to check that  $\dot{f}$  is as desired.

**Theorem 2.8.** *Let  $\omega_2 = S_M \dot{\cup} S_L$ , where the sets  $S_M$  and  $S_L$  are disjoint and stationary. Let  $(P_\alpha, Q_\alpha : \alpha < \omega_2)$  be a countable support iteration of length  $\omega_2$  such that for all  $\alpha$  we have  $\Vdash_{P_\alpha} Q_\alpha = \mathbb{M}$  whenever  $\alpha \in S_M$ , and  $\Vdash_{P_\alpha} Q_\alpha = \mathbb{L}$  otherwise. Also suppose that  $V$  satisfies CH. Then in  $V^P$ ,  $\mathfrak{h} = \omega_1$  holds.*

*Proof.* Both  $\mathbb{M}$  and  $\mathbb{L}$  have the property  $(*)_1$  of [JuSh]. (For  $\mathbb{L}$ , this was proved in [JuSh] and for  $\mathbb{M}$  this was proved in [BaJuSh].) [JuSh] also showed that this property is preserved under countable support iterations, so also  $P_{\omega_2}$  has this property. Hence, the reals of  $V$  do not have measure zero in  $V^P$ , so from  $\mathfrak{h} \leq \mathfrak{s} \leq \text{unif}(\mathfrak{L})$  (where  $\mathfrak{s}$  is the splitting number and  $\text{unif}(\mathfrak{L})$  is the smallest cardinality of a set of reals which is not null) we get the desired conclusion.

**Theorem 2.9.** *Let  $P_{\omega_2}$  be as in Theorem 2.8. Then*

$$\Vdash_{P_{\omega_2}} \omega_1 = \text{add}(I^0) = \text{add}(m^0) < \text{cov}(I^0) = \text{cov}(m^0) = \omega_2.$$

*Proof.* Since  $\mathbb{L}$  adds a dominating real, we have  $\Vdash_{P_{\omega_2}} \mathfrak{b} = \mathfrak{c}$ ; so by Lemma 2.6 and Theorems 2.7 and 2.8 it suffices to prove that the covering coefficients are  $\omega_2$  in the respective models. The proof of this is similar to the proof of [JuMiSh, Theorem 1.2] that  $\text{cov}$  of the Marczewski ideal is  $\omega_2$  in the iterated Sacks's forcing model.

We give the proof only for  $I^0$ . Suppose  $\langle X_\alpha : \alpha < \omega_1 \rangle \in V^{P_{\omega_2}}$  is a sequence of  $I^0$ -sets. In  $V^{P_{\omega_2}}$  let  $f_\alpha : \mathbb{L} \rightarrow \mathbb{L}$  be such that for every  $p \in \mathbb{L}$ ,  $f_\alpha(p)$  extends  $p$  and  $[f_\alpha(p)] \cap X_\alpha = \emptyset$ . Since  $P_{\omega_2}$  has the  $\omega_2$ -chain condition, by a Löwenheim-Skolem argument it is possible to find  $\gamma < \omega_2$  such that

$$\langle f_\alpha \upharpoonright \mathbb{L}^{V_\gamma} : \alpha < \omega_1 \rangle \in V^{P_\gamma}$$

where  $V_\gamma := V^{P_\gamma}$ . Moreover, it is possible to find such a  $\gamma$  in  $S_L$ . We claim that the Laver real  $x_\gamma$  (which is added by  $Q_\gamma = \mathbb{L}^{V_\gamma}$ ) is not in  $\bigcup_{\alpha < \omega_1} X_\alpha$ , which will finish the proof. Otherwise, for some  $p \in \mathbb{L}_{\gamma\omega_2}$  where  $\mathbb{L}_{\gamma\omega_2} := \mathbb{L}_{\omega_2}/G_\gamma$  and some  $\alpha < \omega_1$  we would have  $p \Vdash x_\gamma \in X_\alpha$ . But letting  $q := p(\gamma) \in \mathbb{L}$  and letting  $r(\gamma) := f_\alpha(q)$  and  $r(\beta) := p(\beta)$  for  $\beta > \gamma$  we see that  $r \Vdash x_\gamma \notin X_\alpha$ , a contradiction.

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